Key Terms

Properties of matrix addition
Zero matrix
Properties of matrix multiplication
Properties of scalar multiplication
Properties of transpose

1.4 Exercises

1. Prove Theorem 1.1(b).

2. Prove Theorem 1.1(d).

3. Verify Theorem 1.2(a) for the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix},$$

$$\text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix}.$$

4. Prove Theorem 1.2(b) and (c).

5. Verify Theorem 1.2(c) for the following matrices:

$$A = \begin{bmatrix} 2 & -3 & 2 \\ 3 & -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -2 \end{bmatrix},$$

$$\text{and} \quad C = \begin{bmatrix} 1 & -3 \\ -3 & 4 \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be the $n \times n$ matrix defined by $a_{ii} = k$ and $a_{ij} = 0$ if $i \neq j$. Show that if B is any $n \times n$ matrix, then $AB = kB$.

7. Let A be an $m \times n$ matrix and $C = \begin{bmatrix} c_1 & c_2 & \cdots & c_m \end{bmatrix}$ a $1 \times m$ matrix. Prove that

$$CA = \sum_{j=1}^{m} c_j A_j,$$

where A_j is the jth row of A.

8. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

(a) Determine a simple expression for A^2.

(b) Determine a simple expression for A^3.

(c) Conjecture the form of a simple expression for A^k, k a positive integer.

(d) Prove or disprove your conjecture in part (c).

9. Find a pair of unequal 2×2 matrices A and B, other than those given in Example 9, such that $AB = O$.

10. Find two different 2×2 matrices A such that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

11. Find two unequal 2×2 matrices A and B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

12. Find two different 2×2 matrices A such that $A^2 = O$.

13. Prove Theorem 1.3(a).

14. Prove Theorem 1.3(b).

15. Verify Theorem 1.3(b) for $r = 4$, $s = -2$, and $A = \begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix}$.

16. Prove Theorem 1.3(c).

17. Verify Theorem 1.3(c) for $r = -3$,

$$A = \begin{bmatrix} 4 & 2 \\ 1 & -3 \\ 3 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 4 & 3 \\ -2 & 1 \end{bmatrix}.$$

18. Prove Theorem 1.3(d).

19. Verify Theorem 1.3(d) for the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix},$$

$$\text{and} \quad r = -3.$$

20. The matrix A contains the weight (in pounds) of objects packed on board a spacecraft on earth. The objects are to be used on the moon where things weigh about $\frac{1}{6}$ as much. Write an expression kA that calculates the weight of the objects on the moon.

21. (a) A is a 360×2 matrix. The first column of A is $\cos 0°, \cos 1°, \ldots, \cos 359°$; and the second column is $\sin 0°, \sin 1°, \ldots, \sin 359°$. The graph of the ordered pairs in A is a circle of radius 1 centered at the origin. Write an expression kA for ordered pairs whose graph is a circle of radius 3 centered at the origin.

(b) Explain how to prove the claims about the circles in part (a).

22. Determine a scalar r such that $A\mathbf{x} = r\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

23. Determine a scalar r such that $A\mathbf{x} = r\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ 1 \end{bmatrix}.$$

24. Prove that if $A\mathbf{x} = r\mathbf{x}$ for $n \times n$ matrix A, $n \times 1$ matrix $\mathbf{x}$, and scalar r, then $A\mathbf{y} = r\mathbf{y}$, where $\mathbf{y} = s\mathbf{x}$ for any scalar s.

25. Determine a scalar s such that $A^2\mathbf{x} = s\mathbf{x}$ when $A\mathbf{x} = r\mathbf{x}$.

26. Prove Theorem 1.4(a).

27. Prove Theorem 1.4(b) and (d).

28. Verify Theorem 1.4(a), (b), and (d) for

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & -1 \\ -2 & 1 & 5 \end{bmatrix},$$
$$\text{and} \quad r = -4.$$

29. Verify Theorem 1.4(c) for

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

30. Let

$$A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}.$$

(a) Compute $(AB^T)C$.

(b) Compute B^TC and multiply the result by A on the right. (*Hint*: B^TC is 1×1).

(c) Explain why $(AB^T)C = (B^TC)A$.

31. Determine a constant k such that $(kA)^T(kA) = 1$, where $A = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$. Is there more than one value of k that could be used?

32. Find three 2×2 matrices, A, B, and C such that $AB = AC$ with $B \neq C$ and $A \neq O$.

33. Let A be an $n \times n$ matrix and c a real number. Show that if $cA = O$, then $c = 0$ or $A = O$.

34. Determine all 2×2 matrices A such that $AB = BA$ for any 2×2 matrix B.

35. Show that $(A - B)^T = A^T - B^T$.

36. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

(a) Show that $\mathbf{x}_1 + \mathbf{x}_2$ is a solution.

(b) Show that $\mathbf{x}_1 - \mathbf{x}_2$ is a solution.

(c) For any scalar r, show that $r\mathbf{x}_1$ is a solution.

(d) For any scalars r and s, show that $r\mathbf{x}_1 + s\mathbf{x}_2$ is a solution.

37. Show that if $A\mathbf{x} = \mathbf{b}$ has more than one solution, then it has infinitely many solutions. (*Hint*: If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, consider $\mathbf{x}_3 = r\mathbf{x}_1 + s\mathbf{x}_2$, where $r + s = 1$.)

38. Show that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to the linear system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

39. Let

$$A = \begin{bmatrix} 6 & -1 & 1 \\ 0 & 13 & -16 \\ 0 & 8 & -11 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 10.5 \\ 21.0 \\ 10.5 \end{bmatrix}.$$

(a) Determine a scalar r such that $A\mathbf{x} = r\mathbf{x}$.

(b) Is it true that $A^T\mathbf{x} = r\mathbf{x}$ for the value r determined in part (a)?

40. Repeat Exercise 39 with

$$A = \begin{bmatrix} -3.35 & -3.00 & 3.60 \\ 1.20 & 2.05 & -6.20 \\ -3.60 & -2.40 & 3.85 \end{bmatrix}$$
$$\text{and} \quad \mathbf{x} = \begin{bmatrix} 12.5 \\ -12.5 \\ 6.25 \end{bmatrix}.$$

41. Let $A = \begin{bmatrix} 0.1 & 0.01 \\ 0.001 & 0.0001 \end{bmatrix}$. In your software, set the display format to show as many decimal places as possible, then compute

$$B = 10 * A,$$
$$C = \underbrace{A + A + A + A + A + A + A + A + A + A}_{10 \text{ summands}},$$

and

$$D = B - C.$$

If D is not O, then you have verified that scalar multiplication by a positive integer and successive addition are not the same in your computing environment. (It is not unusual that $D \neq O$, since many computing environments use only a "model" of exact arithmetic, called floating-point arithmetic.)

42. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. In your software, set the display to show as many decimal places as possible. Experiment to find a positive integer k such that $A + 10^{-k} * A$ is equal to A. If you find such an integer k, you have verified that there is more than one matrix in your computational environment that plays the role of O.