Key Terms

Diagonal matrix
Identity matrix
Powers of a matrix
Upper triangular matrix
Lower triangular matrix
Symmetric matrix
Skew symmetric matrix
Submatrix
Partitioning
Partitioned matrix
Nonsingular (invertible) matrix
Inverse
Singular (noninvertible) matrix
Properties of nonsingular matrices
Linear system with nonsingular coefficient matrix
Fibonacci sequence

1.5 Exercises

1. **(a)** Show that if A is any $m \times n$ matrix, then $I_m A = A$ and $AI_n = A$.

(b) Show that if A is an $n \times n$ scalar matrix, then $A = rI_n$ for some real number r.

2. Prove that the sum, product, and scalar multiple of diagonal, scalar, and upper (lower) triangular matrices is diagonal, scalar, and upper (lower) triangular, respectively.

3. Prove: If A and B are $n \times n$ diagonal matrices, then $AB = BA$.

4. Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -3 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

Verify that $A + B$ and AB are upper triangular.

5. Describe all matrices that are both upper and lower triangular.

6. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$. Compute each of the following:

(a) A^2 **(b)** B^3 **(c)** $(AB)^2$

7. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$.

Compute each of the following:

(a) A^3 **(b)** B^2 **(c)** $(AB)^3$

8. Let p and q be nonnegative integers and let A be a square matrix. Show that

$$A^p A^q = A^{p+q} \quad \text{and} \quad (A^p)^q = A^{pq}.$$

9. If $AB = BA$ and p is a nonnegative integer, show that $(AB)^p = A^p B^p$.

10. If p is a nonnegative integer and c is a scalar, show that $(cA)^p = c^p A^p$.

11. For a square matrix A and a nonnegative integer p, show that $(A^T)^p = (A^p)^T$.

12. For a nonsingular matrix A and a nonnegative integer p, show that $(A^p)^{-1} = (A^{-1})^p$.

13. For a nonsingular matrix A and nonzero scalar k, show that $(kA)^{-1} = \frac{1}{k}A^{-1}$.

14. **(a)** Show that every scalar matrix is symmetric.

(b) Is every scalar matrix nonsingular? Explain.

(c) Is every diagonal matrix a scalar matrix? Explain.

15. Find a 2×2 matrix $B \neq O$ and $B \neq I_2$ such that $AB = BA$, where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. How many such matrices B are there?

16. Find a 2×2 matrix $B \neq O$ and $B \neq I_2$ such that $AB = BA$, where $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. How many such matrices B are there?

17. Prove or disprove: For any $n \times n$ matrix A, $A^T A = AA^T$.

18. **(a)** Show that A is symmetric if and only if $a_{ij} = a_{ji}$ for all i, j.

(b) Show that A is skew symmetric if and only if $a_{ij} = -a_{ji}$ for all i, j.

(c) Show that if A is skew symmetric, then the elements on the main diagonal of A are all zero.

19. Show that if A is a symmetric matrix, then A^T is symmetric.

20. Describe all skew symmetric scalar matrices.

21. Show that if A is any $m \times n$ matrix, then AA^T and $A^T A$ are symmetric.

22. Show that if A is any $n \times n$ matrix, then

(a) $A + A^T$ is symmetric.

(b) $A - A^T$ is skew symmetric.

23. Show that if A is a symmetric matrix, then A^k, $k = 2, 3, \ldots$, is symmetric.

24. Let A and B be symmetric matrices.

(a) Show that $A + B$ is symmetric.

(b) Show that AB is symmetric if and only if $AB = BA$.

25. **(a)** Show that if A is an upper triangular matrix, then A^T is lower triangular.

(b) Show that if A is a lower triangular matrix, then A^T is upper triangular.

26. If A is a skew symmetric matrix, what type of matrix is A^T? Justify your answer.

27. Show that if A is skew symmetric, then the elements on the main diagonal of A are all zero.

28. Show that if A is skew symmetric, then A^k is skew symmetric for any positive odd integer k.

29. Show that if A is an $n \times n$ matrix, then $A = S + K$, where S is symmetric and K is skew symmetric. Also show that this decomposition is unique. (*Hint*: Use Exercise 22.)

30. Let

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 6 & 2 \\ 5 & 1 & 3 \end{bmatrix}.$$

Find the matrices S and K described in Exercise 29.

31. Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ is singular.

32. If $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find D^{-1}.

33. Find the inverse of each of the following matrices:

(a) $A = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

34. If A is a nonsingular matrix whose inverse is $\begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$, find A.

35. If

$$A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix},$$

find $(AB)^{-1}$.

36. Suppose that

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Solve the linear system $A\mathbf{x} = \mathbf{b}$ for each of the following matrices $\mathbf{b}$:

(a) $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$

37. The linear system $AC\mathbf{x} = \mathbf{b}$ is such that A and C are nonsingular with

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Find the solution $\mathbf{x}$.

38. The linear system $A^2\mathbf{x} = \mathbf{b}$ is such that A is nonsingular with

$$A^{-1} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Find the solution $\mathbf{x}$.

39. The linear system $A^T\mathbf{x} = \mathbf{b}$ is such that A is nonsingular with

$$A^{-1} = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Find the solution $\mathbf{x}$.

40. The linear system $C^TA\mathbf{x} = \mathbf{b}$ is such that A and C are nonsingular, with

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find the solution $\mathbf{x}$.

41. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where A is the matrix defined in Exercise 33(a).

(a) Find a solution if $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(b) Find a solution if $\mathbf{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

42. Find two 2×2 singular matrices whose sum is nonsingular.

43. Find two 2×2 nonsingular matrices whose sum is singular.

44. Prove Corollary 1.1.

45. Prove Theorem 1.7.

46. Prove that if one row (column) of the $n \times n$ matrix A consists entirely of zeros, then A is singular. (*Hint*: Assume that A is nonsingular; that is, there exists an $n \times n$ matrix B such that $AB = BA = I_n$. Establish a contradiction.)

47. Prove: If A is a diagonal matrix with nonzero diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$, then A is nonsingular and A^{-1} is a diagonal matrix with diagonal entries $1/a_{11}, 1/a_{22}, \ldots, 1/a_{nn}$.

48. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute A^4.

49. For an $n \times n$ diagonal matrix A whose diagonal entries are $a_{11}, a_{22}, \ldots, a_{nn}$, compute A^p for a nonnegative integer p.

50. Show that if $AB = AC$ and A is nonsingular, then $B = C$.