Remark The procedure given here for finding a matrix K in reduced row echelon form that is row equivalent to a given matrix A is not the only one possible. For example, instead of first obtaining a matrix H in row echelon form that is row equivalent to A and then transforming H to reduced row echelon form, we could proceed as follows. First, zero out the entries below a leading 1 and then immediately zero out the entries above the leading 1. This procedure is not as efficient as the procedure given in Example 6.

Key Terms

Elimination method
Reduced row echelon form
Leading one
Row echelon form
Elementary row (column) operation
Row (column) equivalent
Pivot column
Pivot

2.1 Exercises

1. Find a row echelon form of each of the given matrices. Record the row operations you perform, using the notation for elementary row operations.

(a) $A = \begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 2 & -2 & 7 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & -1 \\ 5 & 6 & -3 \\ -2 & -2 & 2 \end{bmatrix}$

2. Find a row echelon form of each of the given matrices. Record the row operations you perform, using the notation for elementary row operations.

(a) $A = \begin{bmatrix} -1 & 1 & -1 & 0 & 3 \\ -3 & 4 & 1 & 1 & 10 \\ 4 & -6 & -4 & -2 & -14 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & -4 \\ -2 & -1 & 10 \\ 4 & 3 & -12 \end{bmatrix}$

3. Each of the given matrices is in row echelon form. Determine its reduced row echelon form. Record the row operations you perform, using the notation for elementary row operations.

(a) $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 4 & 3 & 5 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. Each of the given matrices is in row echelon form. Determine its reduced row echelon form. Record the row operations you perform, using the notation for elementary row operations.

(a) $A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$

5. Find the reduced row echelon form of each of the given matrices. Record the row operations you perform, using the notation for elementary row operations.

(a) $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 9 \\ 3 & 2 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & 0 \\ -2 & 7 & -5 \end{bmatrix}$

6. Find the reduced row echelon form of each of the given matrices. Record the row operations you perform, using the notation for elementary row operations.

(a) $A = \begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 2 & -2 & 7 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & -1 \\ 5 & 6 & -3 \\ -2 & -2 & 2 \end{bmatrix}$

7. Let x, y, z, and w be nonzero real numbers. Label each of the following matrices REF if it is in row echelon form, RREF if it is in reduced row echelon form, or N if it is not REF and not RREF:

(a) $\begin{bmatrix} 1 & x & y & 0 \\ 0 & 1 & 0 & z \\ 0 & 0 & w & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & x & 0 & 0 \\ 0 & 1 & w & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

8. Let x, y, z, and w be nonzero real numbers. Label each of the following matrices REF if it is in row echelon form, RREF if it is in reduced row echelon form, or N if it is not REF and not RREF:

(a) $\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & y & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. Let A be an $n \times n$ matrix in reduced row echelon form. Prove that if $A \neq I_n$, then A has a row consisting entirely of zeros.

10. Prove:

(a) Every matrix is row equivalent to itself.

(b) If B is row equivalent to A, then A is row equivalent to B.

(c) If C is row equivalent to B and B is row equivalent to A, then C is row equivalent to A.

11. Let

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}.$$

(a) Find a matrix in column echelon form that is column equivalent to A.

(b) Find a matrix in reduced column echelon form that is column equivalent to A.

12. Repeat Exercise 11 for the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & -1 & 2 \\ 3 & 1 & 2 & 4 & 1 \end{bmatrix}.$$

13. Determine the reduced row echelon form of

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

2.2 Solving Linear Systems

In this section we use the echelon forms developed in Section 2.1 to more efficiently determine the solution of a linear system compared with the elimination method of Section 1.1. Using the augmented matrix of a linear system together with an echelon form, we develop two methods for solving a system of m linear equations in n unknowns. These methods take the augmented matrix of the linear system, perform elementary row operations on it, and obtain a new matrix that represents an equivalent linear system (i.e., a system that has the same solutions as the original linear system). The important point is that the latter linear system can be solved more easily.

To see how a linear system whose augmented matrix has a particular form can be readily solved, suppose that

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

represents the augmented matrix of a linear system. Then the solution is quickly