Theorem 2.11 If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$. Thus $B = A^{-1}$.

Proof

We first show that if $AB = I_n$, then A is nonsingular. Suppose that A is singular. Then A is row equivalent to a matrix C with a row of zeros. Now $C = E_k E_{k-1} \cdots E_1 A$, where $E_1, E_2, \ldots, E_k$ are elementary matrices. Then $CB = E_k E_{k-1} \cdots E_1 AB$, so AB is row equivalent to CB. Since CB has a row of zeros, we conclude from Theorem 2.10 that AB is singular. Then $AB = I_n$ is impossible, because I_n is nonsingular. This contradiction shows that A is nonsingular, and so A^{-1} exists. Multiplying both sides of the equation $AB = I_n$ by A^{-1} on the left, we then obtain (verify) $B = A^{-1}$. ■

Remark Theorem 2.11 implies that if we want to check whether a given matrix B is A^{-1}, we need merely check whether $AB = I_n$ or $BA = I_n$. That is, we do not have to check both equalities.

Key Terms

Elementary matrix
Inverse matrix
Nonsingular matrix
Singular matrix

2.3 Exercises

1. Prove Theorem 2.5.

2. Let A be a 4×3 matrix. Find the elementary matrix E that, as a premultiplier of A—that is, as EA—performs the following elementary row operations on A:
   (a) Multiplies the second row of A by (-2).
   (b) Adds 3 times the third row of A to the fourth row of A.
   (c) Interchanges the first and third rows of A.

3. Let A be a 3×4 matrix. Find the elementary matrix F that, as a postmultiplier of A—that is, as AF—performs the following elementary column operations on A:
   (a) Adds (-4) times the first column of A to the second column of A.
   (b) Interchanges the second and third columns of A.
   (c) Multiplies the third column of A by 4.

4. Let
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$
   (a) Find a matrix C in reduced row echelon form that is row equivalent to A. Record the row operations used.
   (b) Apply the same operations to I_3 that were used to obtain C. Denote the resulting matrix by B.
   (c) How are A and B related? (*Hint*: Compute AB and BA.)

5. Let
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$
   (a) Find a matrix C in reduced row echelon form that is row equivalent to A. Record the row operations used.
   (b) Apply the same operations to I_3 that were used to obtain C. Denote the resulting matrix by B.
   (c) How are A and B related? (*Hint*: Compute AB and BA.)

6. Prove Theorem 2.7. (*Hint*: Find the inverse of the elementary matrix of type I, type II, and type III.)

7. Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

8. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.

9. Which of the given matrices are singular? For the nonsingular ones, find the inverse.
   (a) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

10. Invert each of the following matrices, if possible:

(a) $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 3 & -3 & -2 \\ 2 & 0 & 1 & 5 \\ 3 & 1 & -2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

11. Find the inverse, if it exists, of each of the following:

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 3 & 3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

12. Find the inverse, if it exists, of each of the following:

(a) $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 0 & 0 \\ 1 & 2 & 1 & -2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{bmatrix}$

In Exercises 13 and 14, prove that each given matrix A is nonsingular and write it as a product of elementary matrices. (Hint: First, write the inverse as a product of elementary matrices; then use Theorem 2.7.)

13. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

15. If A is a nonsingular matrix whose inverse is $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$, find A.

16. If $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, find A.

17. Which of the following homogeneous systems have a nontrivial solution?

(a) $\begin{aligned} x + 2y + 3z &= 0 \\ 2y + 2z &= 0 \\ x + 2y + 3z &= 0 \end{aligned}$

(b) $\begin{aligned} 2x + y - z &= 0 \\ x - 2y - 3z &= 0 \\ -3x - y + 2z &= 0 \end{aligned}$

(c) $\begin{aligned} 3x + y + 3z &= 0 \\ -2x + 2y - 4z &= 0 \\ 2x - 3y + 5z &= 0 \end{aligned}$

18. Which of the following homogeneous systems have a nontrivial solution?

(a) $\begin{aligned} x + y + 2z &= 0 \\ 2x + y + z &= 0 \\ 3x - y + z &= 0 \end{aligned}$

(b) $\begin{aligned} x - y + z &= 0 \\ 2x + y &= 0 \\ 2x - 2y + 2z &= 0 \end{aligned}$

(c) $\begin{aligned} 2x - y + 5z &= 0 \\ 3x + 2y - 3z &= 0 \\ x - y + 4z &= 0 \end{aligned}$

19. Find all value(s) of a for which the inverse of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix}$$

exists. What is A^{-1}?

20. For what values of a does the homogeneous system

$$\begin{aligned} (a-1)x + 2y &= 0 \\ 2x + (a-1)y &= 0 \end{aligned}$$

have a nontrivial solution?

21. Prove that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is nonsingular if and only if $ad - bc \neq 0$. If this condition holds, show that

$$A^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}.$$