Chapter Three

Section 3.1

1. Let $y = e^{rt}$, so that $y' = r\,e^{rt}$ and $y'' = r\,e^{rt}$. Direct substitution into the differential equation yields $(r^2 + 2r - 3)e^{rt} = 0$. Canceling the exponential, the characteristic equation is $r^2 + 2r - 3 = 0$. The roots of the equation are $r = -3, 1$. Hence the general solution is $y = c_1e^t + c_2e^{-3t}$.

2. Let $y = e^{rt}$. Substitution of the assumed solution results in the characteristic equation $r^2 + 3r + 2 = 0$. The roots of the equation are $r = -2, -1$. Hence the general solution is $y = c_1e^{-t} + c_2e^{-2t}$.

4. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $2r^2 - 3r + 1 = 0$. The roots of the equation are $r = 1/2, 1$. Hence the general solution is $y = c_1e^{t/2} + c_2e^t$.

6. The characteristic equation is $4r^2 - 9 = 0$, with roots $r = \pm 3/2$. Therefore the general solution is $y = c_1e^{-3t/2} + c_2e^{3t/2}$.

8. The characteristic equation is $r^2 - 2r - 2 = 0$, with roots $r = 1 \pm \sqrt{3}$. Hence the general solution is $y = c_1 exp\left(1 - \sqrt{3}\right)t + c_2 exp\left(1 + \sqrt{3}\right)t$.

9. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + r - 2 = 0$. The roots of the equation are $r = -2, 1$. Hence the general solution is $y = c_1e^{-2t} + c_2e^t$. Its derivative is $y' = -2c_1e^{-2t} + c_2e^t$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 1$. In order to satisfy $y'(0) = 1$, we find that $-2c_1 + c_2 = 1$. Solving for the constants, $c_1 = 0$ and $c_2 = 1$. Hence the specific solution is $y(t) = e^t$.

11. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $6r^2 - 5r + 1 = 0$. The roots of the equation are $r = 1/3, 1/2$. Hence the general solution is $y = c_1e^{t/3} + c_2e^{t/2}$. Its derivative is $y' = c_1e^{t/3}/3 + c_2e^{t/2}/2$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 4$. In order to satisfy the condition $y'(0) = 1$, we find that $c_1/3 + c_2/2 = 0$. Solving for the constants, $c_1 = 12$ and $c_2 = -8$. Hence the specific solution is $y(t) = 12\,e^{t/3} - 8\,e^{t/2}$.

12. The characteristic equation is $r^2 + 3r = 0$, with roots $r = -3, 0$. Therefore the general solution is $y = c_1 + c_2e^{-3t}$, with derivative $y' = -3\,c_2e^{-3t}$. In order to satisfy the initial conditions, we find that $c_1 + c_2 = -2$, and $-3\,c_2 = 3$. Hence the specific solution is $y(t) = -1 - e^{-3t}$.

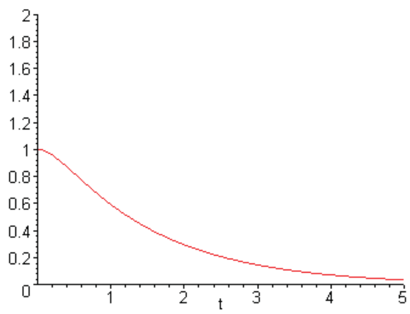
13. The characteristic equation is $r^2 + 5r + 3 = 0$, with roots

$$r_{1,2} = -\frac{5}{2} \pm \frac{\sqrt{13}}{2}\,.$$

The general solution is $y = c_1 exp\Big(-5-\sqrt{13}\Big)t/2 + c_2 exp\Big(-5+\sqrt{13}\Big)t/2$, with derivative

$$y' = \frac{-5-\sqrt{13}}{2}\,c_1 exp\Big(-5-\sqrt{13}\Big)t/2 + \frac{-5+\sqrt{13}}{2}\,c_2 exp\Big(-5+\sqrt{13}\Big)t/2\,.$$

In order to satisfy the initial conditions, we require that $c_1 + c_2 = 1$, and $\frac{-5-\sqrt{13}}{2}\,c_1 + \frac{-5+\sqrt{13}}{2}\,c_2 = 0$. Solving for the coefficients, $c_1 = \Big(1 - 5/\sqrt{13}\Big)/2$ and $c_2 = \Big(1 + 5/\sqrt{13}\Big)/2$.



14. The characteristic equation is $2r^2 + r - 4 = 0$, with roots

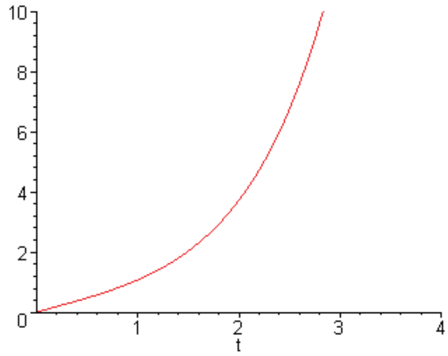
$$r_{1,2} = -\frac{1}{4} \pm \frac{\sqrt{33}}{4}\,.$$

The general solution is $y = c_1 exp\Big(-1-\sqrt{33}\Big)t/4 + c_2 exp\Big(-1+\sqrt{33}\Big)t/4$, with derivative

$$y' = \frac{-1-\sqrt{33}}{4}\,c_1 exp\Big(-1-\sqrt{33}\Big)t/4 + \frac{-1+\sqrt{33}}{4}\,c_2 exp\Big(-1+\sqrt{33}\Big)t/4\,.$$

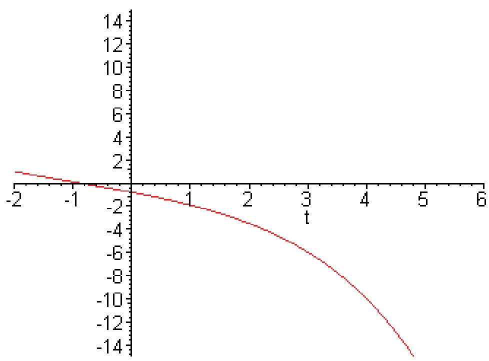
In order to satisfy the initial conditions, we require that $c_1 + c_2 = 0$, and $\frac{-1-\sqrt{33}}{4}\,c_1 + \frac{-1+\sqrt{33}}{4}\,c_2 = 1$. Solving for the coefficients, $c_1 = -2/\sqrt{33}$ and $c_2 = 2/\sqrt{33}$. The specific solution is

$$y(t) = -2\Big[exp\Big(-1-\sqrt{33}\Big)t/4 - exp\Big(-1+\sqrt{33}\Big)t/4\Big]/\sqrt{33}\,.$$



16. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Therefore the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$. Since the initial conditions are specified at $t = -2$, is more convenient to write $y = d_1 e^{-(t+2)/2} + d_2 e^{(t+2)/2}$. The derivative is given by $y' = -\left[d_1 e^{-(t+2)/2}\right]/2 + \left[d_2 e^{(t+2)/2}\right]/2$. In order to satisfy the initial conditions, we find that $d_1 + d_2 = 1$, and $-d_1/2 + d_2/2 = -1$. Solving for the coefficients, $d_1 = 3/2$, and $d_2 = -1/2$. The specific solution is

$$\begin{aligned} y(t) &= \frac{3}{2} e^{-(t+2)/2} - \frac{1}{2} e^{(t+2)/2} \\ &= \frac{3}{2e} e^{-t/2} - \frac{e}{2} e^{t/2} . \end{aligned}$$



18. An algebraic equation with roots -2 and $-1/2$ is $2r^2 + 5r + 2 = 0$. This is the characteristic equation for the ODE $2y'' + 5y' + 2y = 0$.

20. The characteristic equation is $2r^2 - 3r + 1 = 0$, with roots $r = 1/2, 1$. Therefore the general solution is $y = c_1 e^{t/2} + c_2 e^t$, with derivative $y' = c_1 e^{t/2}/2 + c_2 e^t$. In order to satisfy the initial conditions, we require $c_1 + c_2 = 2$ and $c_1/2 + c_2 = 1/2$. Solving for the coefficients, $c_1 = 3$, and $c_2 = -1$. The specific solution is $y(t) = 3e^{t/2} - e^t$. To find the *stationary point*, set $y' = 3e^{t/2}/2 - e^t = 0$. There is a unique solution, with $t_1 = ln(9/4)$. The maximum value is then $y(t_1) = 9/4$. To find