Section 3.6

2. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm 2i$. Hence $y_c(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t$. Since the function $g(t) = 3\, sin\, 2t$ is not proportional to the solutions of the homogeneous equation, set $Y = A\, cos\, 2t + B\, sin\, 2t$. Substitution into the given ODE, and comparing the coefficients, results in the system of equations $B - 4A = 3$ and $A + 4B = 0$. Hence $Y = -\frac{12}{17} cos\, 2t + \frac{3}{17} sin\, 2t$. The general solution is $y(t) = y_c(t) + Y$.

3. The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, with roots $r = -1, 3$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{3t}$. Note that the assignment $Y = Ate^{-t}$ is *not* sufficient to match the coefficients. Try $Y = Ate^{-t} + Bt^2 e^{-t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $-4A + 2B = 0$ and $-8B = -3$. Hence $Y = \frac{3}{16} te^{-t} + \frac{3}{8} t^2 e^{-t}$. The general
solution is $y(t) = y_c(t) + Y$.

5. The characteristic equation for the homogeneous problem is $r^2 + 9 = 0$, with complex roots $r = \pm 3i$. Hence $y_c(t) = c_1 cos\, 3t + c_2 sin\, 3t$. To simplify the analysis, set $g_1(t) = 6$ and $g_2(t) = t^2 e^{3t}$. By inspection, we have $Y_1 = 2/3$. Based on the form of g_2, set $Y_2 = Ae^{3t} + Bte^{3t} + Ct^2 e^{3t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $18A + 6B + 2C = 0$, $18B + 12C = 0$, and $18C = 1$. Hence

$$Y_2 = \frac{1}{162} e^{3t} - \frac{1}{27} te^{3t} + \frac{1}{18} t^2 e^{3t}.$$

The general solution is $y(t) = y_c(t) + Y_1 + Y_2$.

7. The characteristic equation for the homogeneous problem is $2r^2 + 3r + 1 = 0$, with roots $r = -1, -1/2$. Hence $y_c(t) = c_1 e^{-t} + c_2\, e^{-t/2}$. To simplify the analysis, set $g_1(t) = t^2$ and $g_2(t) = 3\, sin\, t$. Based on the form of g_1, set $Y_1 = A + Bt + Ct^2$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $A + 3B + 4C = 0$, $B + 6C = 0$, and $C = 1$. Hence we obtain $Y_1 = 14 - 6t + t^2$. On the other hand, set $Y_2 = D\, cos\, t + E\, sin\, t$. After substitution into the ODE, we find that $D = -9/10$ and $E = -3/10$. The general solution is $y(t) = y_c(t) + Y_1 + Y_2$.

9. The characteristic equation for the homogeneous problem is $r^2 + \omega_0^2 = 0$, with complex roots $r = \pm\, \omega_0 i$. Hence $y_c(t) = c_1 cos\, \omega_0 t + c_2 sin\, \omega_0 t$. Since $\omega \neq \omega_0$, set $Y = A\, cos\, \omega t + B\, sin\, \omega t$. Substitution into the ODE and comparing the coefficients results in the system of equations $(\omega_0^2 - \omega^2)A = 1$ and $(\omega_0^2 - \omega^2)B = 0$. Hence

$$Y = \frac{1}{\omega_0^2 - \omega^2} cos\, \omega t\,.$$

The general solution is $y(t) = y_c(t) + Y$.

10. From Prob. 9, $y_c(t) = c$. Since $cos\,\omega_0 t$ is a solution of the homogeneous problem, set $Y = At\,cos\,\omega_0 t + Bt\,sin\,\omega_0 t$. Substitution into the given ODE and comparing the coefficients results in $A = 0$ and $B = \frac{1}{2\omega_0}$. Hence the general solution is $y(t) = c_1 cos\,\omega_0 t + c_2 sin\,\omega_0 t\, + \frac{t}{2\omega_0} sin\,\omega_0 t$.

12. The characteristic equation for the homogeneous problem is $r^2 - r - 2 = 0$, with roots $r = -1\,, 2$. Hence $y_c(t) = c_1 e^{-t} + c_2\, e^{2t}$. Based on the form of the right hand side, that is, $cosh(2t) = (e^{2t} + e^{-2t})/2$, set $Y = At\, e^{2t} + Be^{-2t}$. Substitution into the given ODE and comparing the coefficients results in $A = 1/6$ and $B = 1/8$. Hence the general solution is $y(t) = c_1 e^{-t} + c_2\, e^{2t} + t\, e^{2t}/6 + e^{-2t}/8$.

14. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm\, 2i$. Hence $y_c(t) = c_1 cos\, 2t + c_2 sin\, 2t$. Set $Y_1 = A + Bt + Ct^2$. Comparing the coefficients of the respective terms, we find that $A = -1/8\,, B = 0\,, C = 1/4$. Now set $Y_2 = D\, e^t$, and obtain $D = 3/5$. Hence the general solution is

$$y(t) = c_1 cos\, 2t + c_2 sin\, 2t - 1/8 + t^2/4 + 3\, e^t/5\,.$$

Invoking the initial conditions, we require that $19/40 + c_1 = 0$ and $3/5 + 2c_2 = 2$. Hence $c_1 = -19/40$ and $c_2 = 7/10$.

15. The characteristic equation for the homogeneous problem is $r^2 - 2r + 1 = 0$, with a double root $r = 1$. Hence $y_c(t) = c_1 e^t + c_2 t\, e^t$. Consider $g_1(t) = t\, e^t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1 = At^2 e^t + Bt^3 e^t$ (the *first* term is not sufficient for a match). Upon substitution, we obtain $Y_1 = t^3 e^t/6$. By inspection, $Y_2 = 4$. Hence the general solution is $y(t) = c_1 e^t + c_2 t\, e^t + t^3 e^t/6\, + 4$. Invoking the initial conditions, we require that $c_1 + 4 = 1$ and $c_1 + c_2 = 1$. Hence $c_1 = -3$ and $c_2 = 4$.

17. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence $y_c(t) = c_1 cos\, 2t + c_2 sin\, 2t$. Since the function $sin\, 2t$ is a solution of the homogeneous problem, set $Y = At\, cos\, 2t + Bt\, sin\, 2t$. Upon substitution, we obtain $Y = -\frac{3}{4} t\, cos\, 2t$. Hence the general solution is $y(t) = c_1 cos\, 2t + c_2 sin\, 2t - \frac{1}{4} t\, cos\, 2t$. Invoking the initial conditions, we require that $c_1 = 2$ and $2c_2 - \frac{3}{4} = -1$. Hence $c_1 = 2$ and $c_2 = -1/8$.

18. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm 2i$. Hence $y_c(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t$. Based on the form of $g(t)$, set $Y = At\, e^{-t} cos\, 2t + Bt\, e^{-t} sin\, 2t$. After comparing coefficients, we obtain $Y = t\, e^{-t} sin\, 2t$. Hence the general solution is

$$y(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t + t\, e^{-t} sin\, 2t\,.$$

Invoking the initial conditions, we require that $c_1 = 1$ and $-c_1 + 2c_2 = 0$. Hence $c_1 = 1$ and $c_2 = 1/2$.

20. The characteristic equation for the homogeneous problem is $r^2+1=0$, with complex roots $r=\pm i$. Hence $y_c(t)=c_1 cos\,t+c_2 sin\,t$. Let $g_1(t)=t\,sin\,t$ and $g_2(t)=t$. By inspection, it is easy to see that $Y_2(t)=1$. Based on the form of $g_1(t)$, set $Y_1(t)=At\,cos\,t+Bt\,sin\,t+Ct^2cos\,t+Dt^2sin\,t$. Substitution into the equation and comparing the coefficients results in $A=0$, $B=1/4$, $C=-1/4$, and $D=0$. Hence $Y(t)=1+\frac{1}{4}t\,sin\,t-\frac{1}{4}t^2cos\,t$.

21. The characteristic equation for the homogeneous problem is $r^2-5r+6=0$, with roots $r=2,3$. Hence $y_c(t)=c_1e^{2t}+c_2e^{3t}$. Consider $g_1(t)=e^{2t}(3t+4)sin\,t$, and $g_2(t)=e^t cos\,2t$. Based on the form of these functions on the right hand side of the ODE,
set $Y_2(t)=e^t(A_1 cos\,2t+A_2 sin\,2t)$, $Y_1(t)=(B_1+B_2t)e^{2t}sin\,t+(C_1+C_2t)e^{2t}cos\,t$. Substitution into the equation and comparing the coefficients results in

$$Y(t)=-\frac{1}{20}\left(e^t cos\,2t+3e^t sin\,2t\right)+\frac{3}{2}te^{2t}(cos\,t-sin\,t)+e^{2t}\left(\frac{1}{2}cos\,t-5sin\,t\right).$$

23. The characteristic roots are $r=2,2$. Hence $y_c(t)=c_1e^{2t}+c_2te^{2t}$. Consider the functions $g_1(t)=2t^2$, $g_2(t)=4te^{2t}$, and $g_3(t)=t\,sin\,2t$. The corresponding forms of the respective parts of the particular solution are $Y_1(t)=A_0+A_1t+A_2t^2$, $Y_2(t)=$ $=e^{2t}(B_2t^2+B_3t^3)$, and $Y_3(t)=t(C_1 cos\,2t+C_2 sin\,2t)+(D_1 cos2t+D_2 sin2t)$. Substitution into the equation and comparing the coefficients results in

$$Y(t)=\frac{1}{4}\left(3+4t+2t^2\right)+\frac{2}{3}t^3e^{2t}+\frac{1}{8}t\,cos\,2t+\frac{1}{16}(cos\,2t-sin\,2t).$$

24. The homogeneous solution is $y_c(t)=c_1 cos\,2t+c_2 sin\,2t$. Since $cos\,2t$ and $sin\,2t$ are both solutions of the homogeneous equation, set

$$Y(t)=t\left(A_0+A_1t+A_2t^2\right)cos\,2t+t\left(B_0+B_1t+B_2t^2\right)sin\,2t.$$

Substitution into the equation and comparing the coefficients results in

$$Y(t)=\left(\frac{13}{32}t-\frac{1}{12}t^3\right)cos\,2t+\frac{1}{16}\left(28t+13t^2\right)sin\,2t.$$

25. The homogeneous solution is $y_c(t)=c_1e^{-t}+c_2te^{-2t}$. None of the functions on the right hand side are solutions of the homogenous equation. In order to include all possible combinations of the derivatives, consider $Y(t)=e^t(A_0+A_1t+A_2t^2)cos\,2t+$ $+e^t(B_0+B_1t+B_2t^2)sin\,2t+e^{-t}(C_1 cos\,t+C_2 sin\,t)+De^t$. Substitution into the differential equation and comparing the coefficients results in

$$Y(t)=e^t\left(A_0+A_1t+A_2t^2\right)cos\,2t+\ +e^t\left(B_0+B_1t+B_2t^2\right)sin\,2t+$$
$$+e^{-t}\left(-\frac{2}{3}cos\,t+\frac{2}{3}sin\,t\right)+2e^t/3,$$