Section 3.7

1. The solution of the homogeneous equation is $y_c(t) = c_1e^{2t} + c_2e^{3t}$. The functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{5t}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{e^{3t}(2e^t)}{W(t)}dt \\ &= 2e^{-t} \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^{2t}(2e^t)}{W(t)}dt \\ &= -e^{-2t} \end{aligned}$$

Hence the particular solution is $Y(t) = 2e^t - e^t = e^t$.

3. The solution of the homogeneous equation is $y_c(t) = c_1e^{-t} + c_2te^{-t}$. The functions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-2t}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^{-t}(3e^{-t})}{W(t)}dt \\ &= -3t^2/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^{-t}(3e^{-t})}{W(t)}dt \\ &= 3t \end{aligned}$$

Hence the particular solution is $Y(t) = -3t^2e^{-t}/2 + 3t^2e^{-t} = 3t^2e^{-t}/2$.

4. The functions $y_1(t) = e^{t/2}$ and $y_2(t) = te^{t/2}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^t$. First write the equation in standard form, so that $g(t) = 4e^{t/2}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^{t/2}\left(4e^{t/2}\right)}{W(t)}dt \\ &= -2t^2 \end{aligned}$$

$$u_2(t) = \int \frac{e^{t/2}\left(4e^{t/2}\right)}{W(t)}dt$$
$$= 4t$$

Hence the particular solution is $Y(t) = -2t^2e^{t/2} + 4t^2e^{t/2} = 2t^2e^{t/2}$.

6. The solution of the homogeneous equation is $y_c(t) = c_1 cos\,3t + c_2 sin\,3t$. The two functions $y_1(t) = cos\,3t$ and $y_2(t) = sin\,3t$ form a fundamental set of solutions, with $W(y_1, y_2) = 3$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{sin\,3t(9\,sec^2 3t)}{W(t)}dt$$
$$= -\,csc\,3t$$

$$u_2(t) = \int \frac{cos\,3t(9\,sec^2 3t)}{W(t)}dt$$
$$= ln|sec\,3t + tan\,3t|$$

Hence the particular solution is $Y(t) = -1 + (sin\,3t)ln|sec\,3t + tan\,3t|$. The general solution is given by $y(t) = c_1 cos\,3t + c_2 sin\,3t + (sin\,3t)ln|sec\,3t + tan\,3t| - 1$.

7. The functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-4t}$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{-2t}(t^{-2}e^{-2t})}{W(t)}dt$$
$$= -\,ln\,t$$

$$u_2(t) = \int \frac{e^{-2t}(t^{-2}e^{-2t})}{W(t)}dt$$
$$= -\,1/t$$

Hence the particular solution is $Y(t) = -\,e^{-2t}ln\,t - e^{-2t}$. Since the *second term* is a solution of the homogeneous equation, the general solution is given by $y(t) = c_1e^{-2t} + + c_2te^{-2t} - e^{-2t}ln\,t$.

8. The solution of the homogeneous equation is $y_c(t) = c_1 cos\,2t + c_2 sin\,2t$. The two functions $y_1(t) = cos\,2t$ and $y_2(t) = sin\,2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{sin\,2t(3\,csc\,2t)}{W(t)}dt \\ &= -\,3t/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{cos\,2t(3\,csc\,2t)}{W(t)}dt \\ &= \frac{3}{4}ln|sin\,2t| \end{aligned}$$

Hence the particular solution is $Y(t) = -\frac{3}{2}tcos\,2t + \frac{3}{4}(sin\,3t)ln|sin\,2t|$. The general solution is given by $y(t) = c_1cos\,2t + c_2sin\,2t - \frac{3}{2}tcos\,2t + \frac{3}{4}(sin\,3t)ln|sin\,2t|$.

9. The functions $y_1(t) = cos\,(t/2)$ and $y_2(t) = sin(t/2)$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = 1/2$. First write the ODE in standard form, so that $g(t) = sec(t/2)/2$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{cos\,(t/2)[sec(t/2)]}{2W(t)}dt \\ &= 2\,ln[cos\,(t/2)] \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{sin(t/2)[sec(t/2)]}{2W(t)}dt \\ &= t \end{aligned}$$

The particular solution is $Y(t) = 2cos(t/2)ln[cos\,(t/2)] + t\,sin(t/2)$. The general solution is given by

$$y(t) = c_1cos\,(t/2) + c_2sin(t/2) + 2\,cos(t/2)\,ln[cos\,(t/2)] + t\,sin(t/2).$$

10. The solution of the homogeneous equation is $y_c(t) = c_1e^t + c_2te^t$. The functions $y_1(t) = e^t$ and $y_2(t) = te^t$ form a fundamental set of solutions, with $W(y_1, y_2) = e^{2t}$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^t(e^t)}{W(t)(1+t^2)}dt \\ &= -\frac{1}{2}ln\big(1+t^2\big) \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^t(e^t)}{W(t)(1+t^2)}dt \\ &= arctan\,t \end{aligned}$$

The particular solution is $Y(t) = -\frac{1}{2}e^t\,ln(1+t^2) + te^t\,arctan(t)$. Hence the general solution is given by $y(t) = c_1e^t + c_2te^t - \frac{1}{2}e^t\, ln(1+t^2) + te^t\, arctan(t)$.

12. The functions $y_1(t) = cos\, 2t$ and $y_2(t) = sin\, 2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)$ $y_1(t) + u_2(t)\, y_2(t)$,
in which

$$u_1(t) = -\frac{1}{2}\int^t g(s)\, sin\, 2s\, ds$$

$$u_2(t) = \frac{1}{2}\int^t g(s)\, cos\, 2s\, ds$$

Hence the particular solution is

$$Y(t) = -\frac{1}{2}cos\, 2t\int^t g(s)\, sin\, 2s\, ds + \frac{1}{2}sin\, 2t\int^t g(s)\, cos\, 2s\, ds\,.$$

Note that $sin\, 2t\, cos\, 2s - cos\, 2t\, sin\, 2s = sin(2t - 2s)$. It follows that

$$Y(t) = \frac{1}{2}\int^t g(s) sin(2t - 2s) ds\,.$$

The general solution of the differential equation is given by

$$y(t) = c_1 cos\, 2t + c_2 sin\, 2t + \frac{1}{2}\int^t g(s) sin(2t - 2s) ds\,.$$

13. Note first that $p(t) = 0\,, q(t) = -2/t^2$ and $g(t) = (3t^2 - 1)/t^2$. The functions $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation, verified by substitution. The Wronskian of these two functions is $W(y_1,y_2) = -3$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{t^{-1}(3t^2 - 1)}{t^2\, W(t)} dt$$
$$= t^{-2}/6 + ln\, t$$

$$u_2(t) = \int \frac{t^2(3t^2 - 1)}{t^2\, W(t)} dt$$
$$= -t^3/3 + t/3$$

Therefore $Y(t) = 1/6 + t^2 ln\, t - t^2/3 + 1/3$. Hence the general solution is

$$y(t) = c_1t^2 + c_2t^{-1} + t^2 ln\, t + 1/2\,.$$

15. Observe that $g(t) = t\,e^{2t}$. The functions $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions. The Wronskian of these two functions is $W(y_1,y_2) = t\,e^t$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{e^t(t\,e^{2t})}{W(t)}dt \\ &= -\,e^{2t}/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{(1+t)(t\,e^{2t})}{W(t)}dt \\ &= t\,e^t \end{aligned}$$

Therefore $Y(t) = -\,(1+t)e^{2t}/2 + t\,e^{2t} = -\,e^{2t}/2 + t\,e^{2t}/2\,.$

16. Observe that $g(t) = 2(1-t)\,e^{-t}$. Direct substitution of $y_1(t) = e^t$ and $y_2(t) = t$ verifies that they are solutions of the homogeneous equation. The Wronskian of the two solutions is $W(y_1,y_2) = (1-t)\,e^t$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{2t(1-t)e^{-t}}{W(t)}dt \\ &= te^{-2t} + e^{-2t}/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{2(1-t)}{W(t)}dt \\ &= -\,2\,e^{-t} \end{aligned}$$

Therefore $Y(t) = te^{-t} + e^{-t}/2 - 2t\,e^{-t} = -\,te^{-t} + \,e^{-t}/2\,.$

17. Note that $g(x) = ln\,x$. The functions $y_1(x) = x^2$ and $y_2(x) = x^2 ln\,x$ are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = x^3$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x),$$

in which

$$\begin{aligned} u_1(x) &= -\int \frac{x^2 ln\,x(ln\,x)}{W(x)}dx \\ &= -\,(ln\,x)^3/3 \end{aligned}$$

$$\begin{aligned} u_2(x) &= \int \frac{x^2(\ln x)}{W(x)} dx \\ &= (\ln x)^2/2 \end{aligned}$$

Therefore $Y(x) = -x^2(\ln x)^3/3 + x^2(\ln x)^3/2 = x^2(\ln x)^3/6$.

19. First write the equation in *standard form*. Note that the forcing function becomes $g(x)/(1-x)$. The functions $y_1(x) = e^x$ and $y_2(x) = x$ are a fundamental set of solutions,
as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = (1-x)e^x$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x),$$

in which

$$u_1(x) = -\int^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau$$

$$u_2(x) = \int^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau$$

Therefore

$$\begin{aligned} Y(x) &= -e^x \int^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau + x\int^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau \\ &= \int^x \frac{(xe^\tau - e^x\tau)g(\tau)}{(1-\tau)^2 e^\tau} d\tau\,. \end{aligned}$$

20. First write the equation in *standard form*. The forcing function becomes $g(x)/x^2$. The functions $y_1(x) = x^{-1/2}\sin x$ and $y_2(x) = x^{-1/2}\cos x$ are a fundamental set of solutions. The Wronskian of the solutions is $W(y_1,y_2) = -1/x$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x),$$

in which

$$u_1(x) = \int^x \frac{\cos\tau\,(g(\tau))}{\tau\sqrt{\tau}} d\tau$$

$$u_2(x) = -\int^x \frac{\sin\tau\,(g(\tau))}{\tau\sqrt{\tau}} d\tau$$

Therefore