Chapter Four

Section 4.1

1. The differential equation is in standard form. Its coefficients, as well as the function $g(t) = t$, are continuous *everywhere*. Hence solutions are valid on the entire real line.

3. Writing the equation in standard form, the coefficients are *rational* functions with singularities at $t = 0$ and $t = 1$. Hence the solutions are valid on the intervals $(-\infty,0)$, $(0,1)$, and $(1,\infty)$.

4. The coefficients are continuous everywhere, but the function $g(t) = ln\, t$ is defined and
continuous only on the interval $(0,\infty)$. Hence solutions are defined for positive reals.

5. Writing the equation in standard form, the coefficients are *rational* functions with a singularity at $x_0 = 1$. Furthermore, $p_4(x) = tan\, x/(x-1)$ is *undefined*, and hence not continuous, at $x_k = \pm(2k+1)\pi/2$, $k = 0, 1, 2, \cdots$. Hence solutions are defined on any *interval* that *does not* contain x_0 or x_k.

6. Writing the equation in standard form, the coefficients are *rational* functions with singularities at $x = \pm 2$. Hence the solutions are valid on the intervals $(-\infty, -2)$, $(-2,2)$, and $(2,\infty)$.

7. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = -14$. Hence the functions are linearly *independent*.

9. Evaluating the Wronskian of the four functions, $W(f_1, f_2, f_3, f_4) = 0$. Hence the functions are linearly *dependent*. To find a linear relation among the functions, we need to find constants c_1, c_2, c_3, c_4, not all zero, such that

$$c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) + c_4 f_4(t) = 0\,.$$

Collecting the common terms, we obtain

$$(c_2 + 2c_3 + c_4)t^2 + (2c_1 - c_3 + c_4)t + (-3c_1 + c_2 + c_4) = 0\,,$$

which results in *three* equations in *four* unknowns. Arbitrarily setting $c_4 = -1$, we can solve the equations $c_2 + 2c_3 = 1$, $2c_1 - c_3 = 1$, $-3c_1 + c_2 = 1$, to find that $c_1 = 2/7$, $c_2 = 13/7$, $c_3 = -3/7$. Hence

$$2f_1(t) + 13f_2(t) - 3f_3(t) - 7f_4(t) = 0\,.$$

10. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = 156$. Hence the functions are linearly *independent*.

11. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have
$W(1, cos\, t, sin\, t) = 1$.

12. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, t, cos\, t, sin\, t) = 1$.

14. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, t, e^{-t}, t\, e^{-t}) = e^{-2t}$.

15. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, x, x^3) = 6x$.

16. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(x, x^2, 1/x) = 6/x$.

18. The operation of taking a derivative is linear, and hence

$$(c_1 y_1 + c_2 y_2)^{(k)} = c_1 y_1^{(k)} + c_2 y_2^{(k)}.$$

It follows that

$$L[c_1 y_1 + c_2 y_2] = c_1 y_1^{(n)} + c_2 y_2^{(n)} + p_1 \left[c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)}\right] + \cdots + p_n [c_1 y_1 + c_2 y_2].$$

Rearranging the terms, we obtain $L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$. Since y_1 and y_2 are solutions, $L[c_1 y_1 + c_2 y_2] = 0$. The rest follows by induction.

19(a). Note that $d^k(t^n)/dt^k = n(n-1)\cdots(n-k+1)t^{n-k}$, for $k = 1, 2, \cdots, n$. Hence

$$L[t^n] = a_0\, n! + a_1[n(n-1)\cdots 2]t + \cdots a_{n-1}\, n\, t^{n-1} + a_n\, t^n.$$

(b). We have $d^k(e^{rt})/dt^k = r^k e^{rt}$, for $k = 0, 1, 2, \cdots$. Hence

$$\begin{aligned} L\left[e^{rt}\right] &= a_0\, r^n e^{rt} + a_1 r^{n-1} e^{rt} + \cdots a_{n-1} r\, e^{rt} + a_n\, e^{rt} \\ &= \left[a_0\, r^n + a_1 r^{n-1} + \cdots a_{n-1}\, r\, + a_n\right] e^{rt}. \end{aligned}$$

(c). Set $y = e^{rt}$, and substitute into the ODE. It follows that $r^4 - 5r^2 + 4 = 0$, with $r = \pm 1, \pm 2$. Furthermore, $W(e^t, e^{-t}, e^{2t}, e^{-2t}) = 72$.

20(a). Let $f(t)$ and $g(t)$ be arbitrary functions. Then $W(f, g) = fg' - f'g$. Hence $W'(f, g) = f'g' + fg'' - f''g - f'g' = fg'' - f''g$. That is,

$$W'(f, g) = \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix}.$$

Now expand the 3-*by*-3 determinant as