equation can be expressed as a linear combination of a fundamental set of solutions $y_1, \dots, y_n$, it follows that any solution of Eq. (2) can be written as

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t), \tag{9}$$

where Y is some particular solution of the nonhomogeneous equation (2). The linear combination (9) is called the general solution of the nonhomogeneous equation (2).

Thus the primary problem is to determine a fundamental set of solutions $y_1, \dots, y_n$ of the homogeneous equation (4). If the coefficients are constants, this is a fairly simple problem; it is discussed in the next section. If the coefficients are not constants, it is usually necessary to use numerical methods such as those in Chapter 8 or series methods similar to those in Chapter 5. These tend to become more cumbersome as the order of the equation increases.

The method of reduction of order (Section 3.5) also applies to nth order linear equations. If y_1 is one solution of Eq. (4), then the substitution $y = v(t)y_1(t)$ leads to a linear differential equation of order $n-1$ for v' (see Problem 26 for the case when $n = 3$). However, if $n \geq 3$, the reduced equation is itself at least of second order, and only rarely will it be significantly simpler than the original equation. Thus, in practice, reduction of order is seldom useful for equations of higher than second order.

PROBLEMS

In each of Problems 1 through 6 determine intervals in which solutions are sure to exist.

1. $y^{\text{iv}} + 4y''' + 3y = t$
2. $ty''' + (\sin t)y'' + 3y = \cos t$
3. $t(t-1)y^{\text{iv}} + e^t y'' + 4t^2 y = 0$
4. $y''' + ty'' + t^2 y' + t^3 y = \ln t$
5. $(x-1)y^{\text{iv}} + (x+1)y'' + (\tan x)y = 0$
6. $(x^2-4)y^{\text{vi}} + x^2 y''' + 9y = 0$

In each of Problems 7 through 10 determine whether the given set of functions is linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

7. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t$
8. $f_1(t) = 2t - 3, \quad f_2(t) = 2t^2 + 1, \quad f_3(t) = 3t^2 + t$
9. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$
10. $f_1(t) = 2t - 3, \quad f_2(t) = t^3 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$

In each of Problems 11 through 16 verify that the given functions are solutions of the differential equation, and determine their Wronskian.

11. $y''' + y' = 0; \qquad 1, \quad \cos t, \quad \sin t$
12. $y^{\text{iv}} + y'' = 0; \qquad 1, \quad t, \quad \cos t, \quad \sin t$
13. $y''' + 2y'' - y' - 2y = 0; \qquad e^t, \quad e^{-t}, \quad e^{-2t}$
14. $y^{\text{iv}} + 2y''' + y'' = 0; \qquad 1, \quad t, \quad e^{-t}, \quad te^{-t}$
15. $xy''' - y'' = 0; \qquad 1, \quad x, \quad x^3$
16. $x^3 y''' + x^2 y'' - 2xy' + 2y = 0; \qquad x, \quad x^2, \quad 1/x$
17. Show that $W(5, \sin^2 t, \cos 2t) = 0$ for all t. Can you establish this result without direct evaluation of the Wronskian?
18. Verify that the differential operator defined by

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y$$

is a linear differential operator. That is, show that

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2],$$