Section 4.3

2. The general solution of the homogeneous equation is $y_c = c_1e^t + c_2e^{-t} + c_3cos\,t + $ $+ c_4sin\,t$. Let $g_1(t) = 3t$ and $g_2(t) = cos\,t$. By inspection, we find that $Y_1(t) = -3t$. Since $g_2(t)$ is a solution of the homogeneous equation, set $Y_2(t) = t(Acos\,t + Bsin\,t)$. Substitution into the given ODE and comparing the coefficients of similar term results in $A = 0$ and $B = -1/4$. Hence the general solution of the nonhomogeneous problem is

$$y(t) = y_c(t) - 3t - \frac{t}{4}sin\,t\,.$$

3. The characteristic equation corresponding to the homogeneous problem can be written as $(r+1)(r^2+1) = 0$. The solution of the homogeneous equation is $y_c = c_1e^{-t} +$ $+ c_2cos\,t + c_3sin\,t$. Let $g_1(t) = e^{-t}$ and $g_2(t) = 4t$. Since $g_1(t)$ is a solution of the homogeneous equation, set $Y_1(t) = Ate^{-t}$. Substitution into the ODE results in $A = 1/2$. Now let $Y_2(t) = Bt + C$. We find that $B = -C = 4$. Hence the general solution of the nonhomogeneous problem is $y(t) = y_c(t) + te^{-t}/2 + 4(t-1)$.

4. The characteristic equation corresponding to the homogeneous problem can be written as $r(r+1)(r-1) = 0$. The solution of the homogeneous equation is $y_c = c_1 + c_2e^t +$ $+ c_3e^{-t}$. Since $g(t) = 2\,sin\,t$ is not a solution of the homogeneous problem, we can set $Y(t) = A\,cos\,t + B\,sin\,t$. Substitution into the ODE results in $A = 1$ and $B = 0$. Thus
the general solution is $y(t) = c_1 + c_2e^t + c_3e^{-t} + cos\,t$.

6. The characteristic equation corresponding to the homogeneous problem can be written as $(r^2+1)^2 = 0$. It follows that $y_c = c_1cos\,t + c_2sin\,t + t(c_3cos\,t + c_4sin\,t)$. Since $g(t)$ is not a solution of the homogeneous problem, set $Y(t) = A + Bcos\,2t + Csin\,2t$. Substitution into the ODE results in $A = 3,\ B = 1/9,\ C = 0$. Thus the general solution is $y(t) = y_c(t) + 3 + \frac{1}{9}cos\,2t$.

7. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r^3+1) = 0$. Thus the homogeneous solution is

$$y_c = c_1 + c_2\,t + c_3t^2 + c_4e^{-t} + e^{t/2}\Big[c_5cos\Big(\sqrt{3}\,t/2\Big) + c_5sin\Big(\sqrt{3}\,t/2\Big)\Big].$$

Note the $g(t) = t$ is a solution of the homogenous problem. Consider a particular solution
of the form $Y(t) = t^3(At + B)$. Substitution into the ODE results in $A = 1/24$ and $B = 0$. Thus the general solution is $y(t) = y_c(t) + t^4/24$.

8. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r+1) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2\,t + c_3t^2 + c_4e^{-t}$. Since $g(t)$ is *not* a solution of the homogeneous problem, set $Y(t) = Acos\,2t + Bsin\,2t$. Substitution into the ODE results in $A = 1/40$ and $B = 1/20$. Thus the general solution

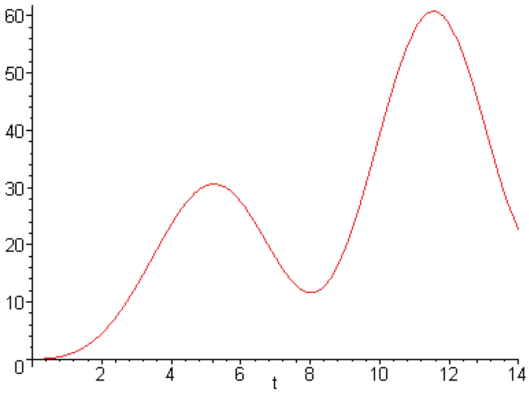
is $y(t) = y_c(t) + (cos\, 2t + 2sin\, 2t)/40$.

10. From Prob. 22 in Section 4.2 , the homogeneous solution is

$$y_c = c_1 cos\, t + c_2 sin\, t \; + \; + t[c_3 cos\, t + c_4 sin\, t].$$

Since $g(t)$ is *not* a solution of the homogeneous problem, substitute $Y(t) = At + B$ into the ODE to obtain $A = 3$ and $B = 4$. Thus the general solution is $y(t) = y_c(t) + 3t + 4$. Invoking the initial conditions, we find that $c_1 = -4$, $c_2 = -4$, $c_3 = 1$, $c_4 = -3/2$. Therefore the solution of the initial value problem is

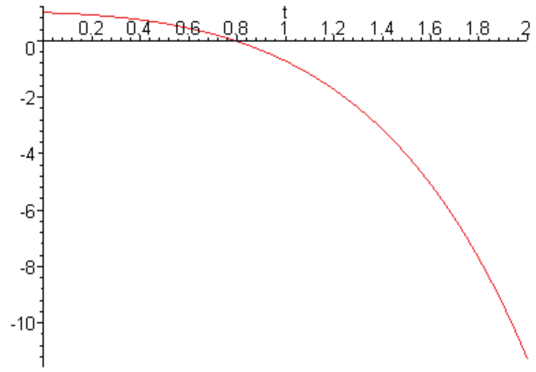
$$y(t) = (t - 4)cos\, t - (3t/2 + 4)sin\, t + 3t + 4\,.$$



11. The characteristic equation can be written as $r(r^2 - 3r + 2) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2 e^t + c_3 e^{2t}$. Let $g_1(t) = e^t$ and $g_2(t) = t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1(t) = Ate^t$. Substitution into the ODE results in $A = -1$. Now let $Y_2(t) = Bt^2 + Ct$. Substitution into the ODE results in $B = 1/4$ and $C = 3/4$. Therefore the general solution is

$$y(t) = c_1 + c_2 e^t + c_3 e^{2t} - te^t + \left(t^2 + 3t\right)/4\,.$$

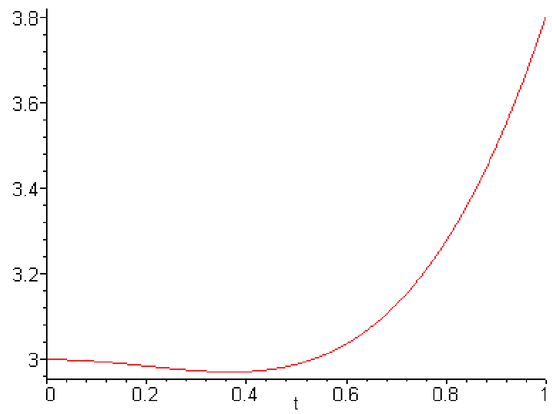
Invoking the initial conditions, we find that $c_1 = 1$, $c_2 = c_3 = 0$. The solution of the initial value problem is $y(t) = 1 - te^t + (t^2 + 3t)/4$.



12. The characteristic equation can be written as $(r-1)(r+3)(r^2+4)=0$. Hence the homogeneous solution is $y_c = c_1 e^t + c_2 e^{-3t} + c_3 cos\, 2t + c_4 sin\, 2t$. None of the terms in $g(t)$ is a solution of the homogeneous problem. Therefore we can assume a form $Y(t) = Ae^{-t} + Bcos\, t + Csin\, t$. Substitution into the ODE results in $A = 1/20$, $B = -2/5$, $C = -4/5$. Hence the general solution is

$$y(t) = c_1 e^t + c_2 e^{-3t} + c_3 cos\, 2t + c_4 sin\, 2t + e^{-t}/20 - (2cos\, t + 4sin\, t)/5\,.$$

Invoking the initial conditions, we find that $c_1 = 81/40$, $c_2 = 73/520$, $c_3 = 77/65$, $c_4 = -49/130$.



14. From Prob. 4, the homogeneous solution is $y_c = c_1 + c_2 e^t + c_3 e^{-t}$. Consider the terms $g_1(t) = te^{-t}$ and $g_2(t) = 2cos\, t$. Note that since $r = -1$ is a *simple* root of the characteristic equation, Table 4.3.1 suggests that we set $Y_1(t) = t(At+B)e^{-t}$. The function $2cos\, t$ is *not* a solution of the homogeneous equation. We can simply choose $Y_2(t) = Ccos\, t + Dsin\, t$. Hence the particular solution has the form

$$Y(t) = t(At+B)e^{-t} + Ccos\, t + Dsin\, t\,.$$

15. The characteristic equation can be written as $(r^2-1)^2 = 0$. The roots are given

as $r = \pm 1$, each with *multiplicity two*. Hence the solution of the homogeneous problem is $y_c = c_1 e^t + c_2 te^t + c_3 e^{-t} + c_4 te^{-t}$. Let $g_1(t) = e^t$ and $g_2(t) = sin\, t$. The function e^t is a solution of the homogeneous problem. Since $r = 1$ has multiplicity two, we set $Y_1(t) = At^2 e^t$. The function $sin\, t$ is *not* a solution of the homogeneous equation. We can set $Y_2(t) = Bcos\, t + Csin\, t$. Hence the particular solution has the form

$$Y(t) = At^2 e^t + Bcos\, t + Csin\, t\,.$$

16. The characteristic equation can be written as $r^2(r^2 + 4) = 0$, with roots $r = 0, \pm 2i$. The root $r = 0$ has multiplicity two, hence the homogeneous solution is $y_c = c_1 + c_2 t +$ $+ c_3 cos\, 2t + c_4 sin\, 2t$. The functions $g_1(t) = sin\, 2t$ and $g_2(t) = 4$ are solutions of the homogenous equation. The complex roots have multiplicity *one*, therefore we need to set $Y_1(t) = At\, cos\, 2t + Bt\, sin\, 2t$. Now $g_2(t) = 4$ is associated with the *double* root $r = 0$. Based on Table 4.3.1, set $Y_2(t) = Ct^2$. Finally, $g_3(t) = te^t$ (and its derivatives) is independent of the homogeneous solution. Therefore set $Y_3(t) = (Dt + E)e^t$. Conclude that the particular solution has the form

$$Y(t) = At\, cos\, 2t + Bt\, sin\, 2t + Ct^2 + (Dt + E)e^t.$$

18. The characteristic equation can be written as $r^2(r^2 + 2r + 2) = 0$, with roots $r = 0$, with multiplicity *two*, and $r = -1 \pm i$. The homogeneous solution is $y_c = c_1 + c_2 t +$ $+ c_3 e^{-t} cos\, t + c_4 e^{-t} sin\, t$. The function $g_1(t) = 3e^t + 2te^{-t}$, and all of its derivatives, is independent of the homogeneous solution. Therefore set $Y_1(t) = Ae^t + (Bt + C)e^{-t}$. Now $g_2(t) = e^{-t} sin\, t$ is a solution of the homogeneous equation, associated with the complex roots. We need to set $Y_2(t) = t(D\, e^{-t} cos\, t + E\, e^{-t} sin\, t)$. It follows that the particular solution has the form

$$Y(t) = Ae^t + (Bt + C)e^{-t} + t\big(D\, e^{-t} cos\, t + E\, e^{-t} sin\, t\big).$$

19. Differentiating $y = u(t)v(t)$, successively, we have

$$\begin{aligned} y' &= u'v + uv' \\ y'' &= u''v + 2u'v' + uv'' \\ &\vdots \\ y^{(n)} &= \sum_{j=0}^{n} \binom{n}{j} u^{(n-j)} v^{(j)} \end{aligned}$$

Setting $v(t) = e^{\alpha t}$, $v^{(j)} = \alpha^j e^{\alpha t}$. So for any $p = 1, 2, \cdots, n$,

$$y^{(p)} = e^{\alpha t} \sum_{j=0}^{p} \binom{p}{j} \alpha^j\, u^{(p-j)}.$$

It follows that