We can write a particular solution of Eq. (8) as the sum of particular solutions of the differential equations

$$y''' - 4y' = t, \qquad y''' - 4y' = 3\cos t, \qquad y''' - 4y' = e^{-2t}.$$

Our initial choice for a particular solution $Y_1(t)$ of the first equation is $A_0 t + A_1$, but since a constant is a solution of the homogeneous equation, we multiply by t. Thus

$$Y_1(t) = t(A_0 t + A_1).$$

For the second equation we choose

$$Y_2(t) = B\cos t + C\sin t,$$

and there is no need to modify this initial choice since $\cos t$ and $\sin t$ are not solutions of the homogeneous equation. Finally, for the third equation, since e^{-2t} is a solution of the homogeneous equation, we assume that

$$Y_3(t) = Ete^{-2t}.$$

The constants are determined by substituting into the individual differential equations; they are $A_0 = -\frac{1}{8}$, $A_1 = 0$, $B = 0$, $C = -\frac{3}{5}$, and $E = \frac{1}{8}$. Hence a particular solution of Eq. (8) is

$$Y(t) = -\tfrac{1}{8}t^2 - \tfrac{3}{5}\sin t + \tfrac{1}{8}te^{-2t}. \tag{9}$$

The amount of algebra required to calculate the coefficients may be quite substantial for higher order equations, especially if the nonhomogeneous term is even moderately complicated. A computer algebra system can be extremely helpful in executing these algebraic calculations.

The method of undetermined coefficients can be used whenever it is possible to guess the correct form for $Y(t)$. However, this is usually impossible for differential equations not having constant coefficients, or for nonhomogeneous terms other than the type described previously. For more complicated problems we can use the method of variation of parameters, which is discussed in the next section.

PROBLEMS

In each of Problems 1 through 8 determine the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 2e^{-t} + 3$
2. $y^{\text{iv}} - y = 3t + \cos t$
3. $y''' + y'' + y' + y = e^{-t} + 4t$
4. $y''' - y' = 2\sin t$
5. $y^{\text{iv}} - 4y'' = t^2 + e^t$
6. $y^{\text{iv}} + 2y'' + y = 3 + \cos 2t$
7. $y^{\text{vi}} + y''' = t$
8. $y^{\text{iv}} + y''' = \sin 2t$

In each of Problems 9 through 12 find the solution of the given initial value problem. Then plot a graph of the solution.

9. $y''' + 4y' = t, \quad y(0) = y'(0) = 0, \quad y''(0) = 1$
10. $y^{\text{iv}} + 2y'' + y = 3t + 4, \quad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1$
11. $y''' - 3y'' + 2y' = t + e^t, \quad y(0) = 1, \quad y'(0) = -\frac{1}{4}, \quad y''(0) = -\frac{3}{2}$
12. $y^{\text{iv}} + 2y''' + y'' + 8y' - 12y = 12\sin t - e^{-t}, \quad y(0) = 3, \quad y'(0) = 0,$
    $y''(0) = -1, \quad y'''(0) = 2$

In each of Problems 13 through 18 determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

13. $y''' - 2y'' + y' = t^3 + 2e^t$
14. $y''' - y' = te^{-t} + 2\cos t$
15. $y^{\text{iv}} - 2y'' + y = e^t + \sin t$
16. $y^{\text{iv}} + 4y'' = \sin 2t + te^t + 4$
17. $y^{\text{iv}} - y''' - y'' + y' = t^2 + 4 + t\sin t$
18. $y^{\text{iv}} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$
19. Consider the nonhomogeneous nth order linear differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = g(t),$$

where $a_0, \ldots, a_n$ are constants. Verify that if $g(t)$ is of the form

$$e^{\alpha t}(b_0 t^m + \cdots + b_m),$$

then the substitution $y = e^{\alpha t}u(t)$ reduces the preceding equation to the form

$$k_0 u^{(n)} + k_1 u^{(n-1)} + \cdots + k_n u = b_0 t^m + \cdots + b_m,$$

where $k_0, \ldots, k_n$ are constants. Determine k_0 and k_n in terms of the a's and α. Thus the problem of determining a particular solution of the original equation is reduced to the simpler problem of determining a particular solution of an equation with constant coefficients and a polynomial for the nonhomogeneous term.

Method of Annihilators. In Problems 20 through 22 we consider another way of arriving at the proper form of $Y(t)$ for use in the method of undetermined coefficients. The procedure is based on the observation that exponential, polynomial, or sinusoidal terms (or sums and products of such terms) can be viewed as solutions of certain linear homogeneous differential equations with constant coefficients. It is convenient to use the symbol D for d/dt. Then, for example, e^{-t} is a solution of $(D+1)y = 0$; the differential operator $D+1$ is said to *annihilate*, or to be an *annihilator* of, e^{-t}. Similarly, $D^2 + 4$ is an annihilator of $\sin 2t$ or $\cos 2t$, $(D-3)^2 = D^2 - 6D + 9$ is an annihilator of e^{3t} or te^{3t}, and so forth.

20. Show that linear differential operators with constant coefficients obey the commutative law, that is,

$$(D-a)(D-b)f = (D-b)(D-a)f$$

for any twice differentiable function f and any constants a and b. The result extends at once to any finite number of factors.

21. Consider the problem of finding the form of the particular solution $Y(t)$ of

$$(D-2)^3(D+1)Y = 3e^{2t} - te^{-t}, \qquad \text{(i)}$$

where the left side of the equation is written in a form corresponding to the factorization of the characteristic polynomial.
(a) Show that $D-2$ and $(D+1)^2$, respectively, are annihilators of the terms on the right side of Eq. (i), and that the combined operator $(D-2)(D+1)^2$ annihilates both terms on the right side of Eq. (i) simultaneously.
(b) Apply the operator $(D-2)(D+1)^2$ to Eq. (i) and use the result of Problem 20 to obtain

$$(D-2)^4(D+1)^3 Y = 0. \qquad \text{(ii)}$$

Thus Y is a solution of the homogeneous equation (ii). By solving Eq. (ii), show that

$$Y(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 t^2 e^{2t} + c_4 t^3 e^{2t} + c_5 e^{-t} + c_6 te^{-t} + c_7 t^2 e^{-t}, \qquad \text{(iii)}$$

where $c_1, \ldots, c_7$ are constants, as yet undetermined.