

## EXERCISES 5.6

### Evaluating Definite Integrals

Use the Substitution Formula in Theorem 6 to evaluate the integrals in Exercises 1–24.

1. a.  $\int_0^3 \sqrt{y+1} \, dy$

b.  $\int_{-1}^0 \sqrt{y+1} \, dy$

2. a.  $\int_0^1 r\sqrt{1-r^2} \, dr$

b.  $\int_{-1}^1 r\sqrt{1-r^2} \, dr$

3. a.  $\int_0^{\pi/4} \tan x \sec^2 x \, dx$

b.  $\int_{-\pi/4}^0 \tan x \sec^2 x \, dx$

4. a.  $\int_0^{\pi} 3 \cos^2 x \sin x \, dx$

b.  $\int_{2\pi}^{3\pi} 3 \cos^2 x \sin x \, dx$

5. a.  $\int_0^1 t^3(1+t^4)^3 \, dt$

b.  $\int_{-1}^1 t^3(1+t^4)^3 \, dt$

6. a.  $\int_0^{\sqrt{7}} t(t^2+1)^{1/3} \, dt$

b.  $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3} \, dt$

7. a.  $\int_{-1}^1 \frac{5r}{(4+r^2)^2} \, dr$

b.  $\int_0^1 \frac{5r}{(4+r^2)^2} \, dr$

8. a.  $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2} \, dv$

b.  $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2} \, dv$

9. a.  $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} \, dx$

b.  $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} \, dx$

10. a.  $\int_0^1 \frac{x^3}{\sqrt{x^4+9}} \, dx$

b.  $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}} \, dx$

11. a.  $\int_0^{\pi/6} (1 - \cos 3t) \sin 3t \, dt$

b.  $\int_{-\pi/6}^{\pi/3} (1 - \cos 3t) \sin 3t \, dt$

12. a.  $\int_{-\pi/2}^0 \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2} \, dt$

b.  $\int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2} \, dt$

13. a.  $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}} \, dz$

b.  $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}} \, dz$

14. a.  $\int_{-\pi/2}^0 \frac{\sin w}{(3+2\cos w)^2} \, dw$

b.  $\int_0^{\pi/2} \frac{\sin w}{(3+2\cos w)^2} \, dw$

15.  $\int_0^1 \sqrt{t^5+2t}(5t^4+2) \, dt$

16.  $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

17.  $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta \, d\theta$

18.  $\int_{\pi}^{3\pi/2} \cot^5 \left(\frac{\theta}{6}\right) \sec^2 \left(\frac{\theta}{6}\right) \, d\theta$

19.  $\int_0^{\pi} 5(5-4\cos t)^{1/4} \sin t \, dt$

20.  $\int_0^{\pi/4} (1-\sin 2t)^{3/2} \cos 2t \, dt$

21.  $\int_0^1 (4y-y^2+4y^3+1)^{-2/3} (12y^2-2y+4) \, dy$

22.  $\int_0^1 (y^3+6y^2-12y+9)^{-1/2} (y^2+4y-4) \, dy$

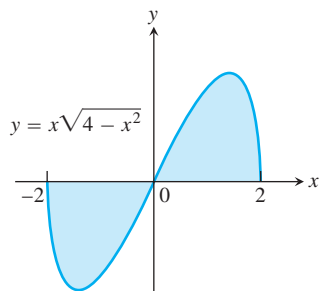
23.  $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta} \cos^2(\theta^{3/2}) d\theta$

24.  $\int_{-1}^{-1/2} t^{-2} \sin^2\left(1 + \frac{1}{t}\right) dt$

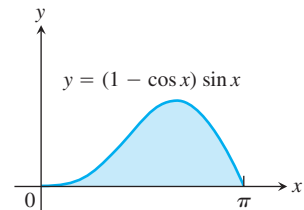
**Area**

Find the total areas of the shaded regions in Exercises 25–40.

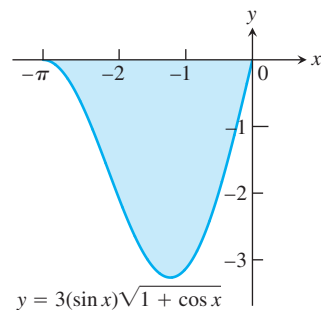
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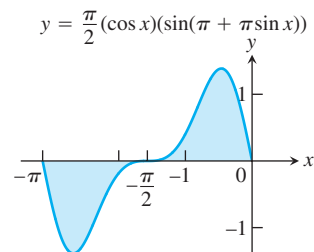
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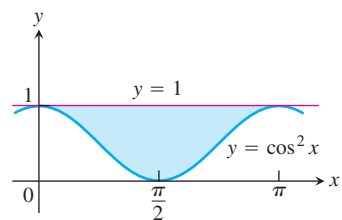
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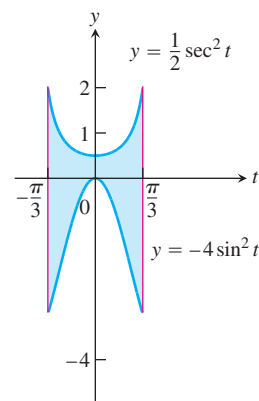
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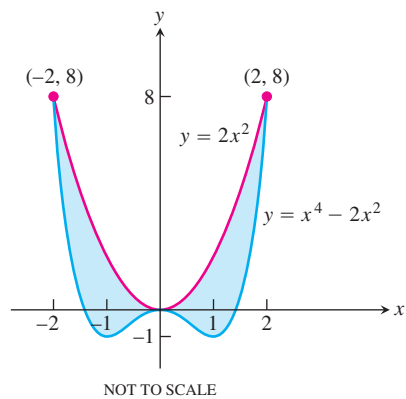
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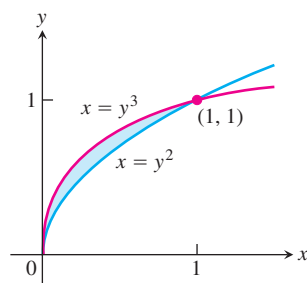
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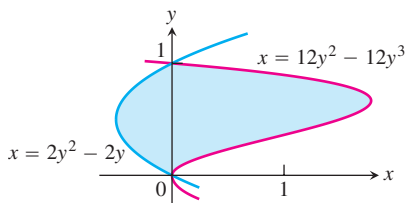
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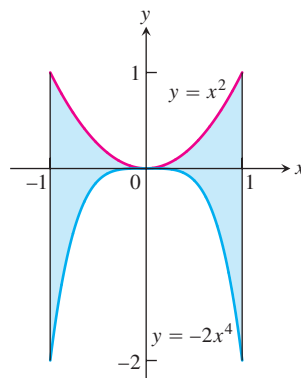
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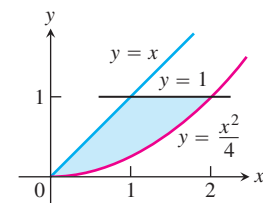
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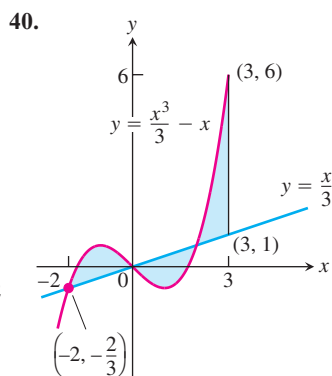
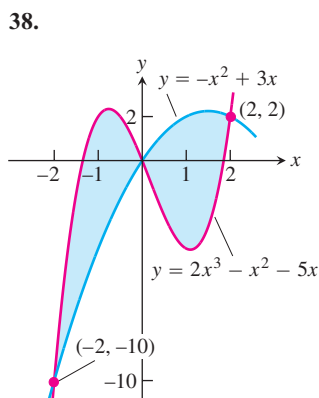
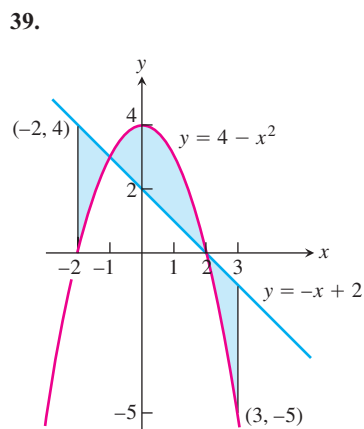
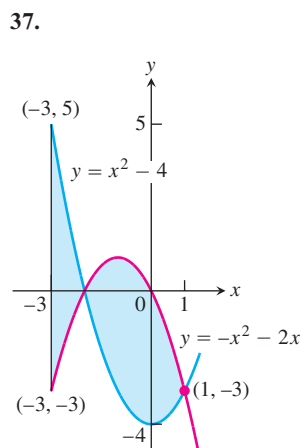
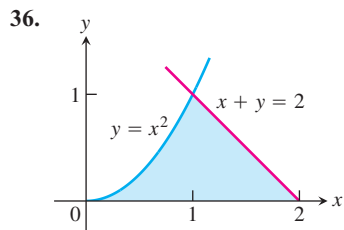


34.



35.





Find the areas of the regions enclosed by the lines and curves in Exercises 41–50.

41.  $y = x^2 - 2$  and  $y = 2$
42.  $y = 2x - x^2$  and  $y = -3$
43.  $y = x^4$  and  $y = 8x$
44.  $y = x^2 - 2x$  and  $y = x$
45.  $y = x^2$  and  $y = -x^2 + 4x$
46.  $y = 7 - 2x^2$  and  $y = x^2 + 4$
47.  $y = x^4 - 4x^2 + 4$  and  $y = x^2$
48.  $y = x\sqrt{a^2 - x^2}$ ,  $a > 0$ , and  $y = 0$

49.  $y = \sqrt{|x|}$  and  $5y = x + 6$  (How many intersection points are there?)

50.  $y = |x^2 - 4|$  and  $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 51–58.

51.  $x = 2y^2$ ,  $x = 0$ , and  $y = 3$

52.  $x = y^2$  and  $x = y + 2$

53.  $y^2 - 4x = 4$  and  $4x - y = 16$

54.  $x - y^2 = 0$  and  $x + 2y^2 = 3$

55.  $x + y^2 = 0$  and  $x + 3y^2 = 2$

56.  $x - y^{2/3} = 0$  and  $x + y^4 = 2$

57.  $x = y^2 - 1$  and  $x = |y|\sqrt{1 - y^2}$

58.  $x = y^3 - y^2$  and  $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 59–62.

59.  $4x^2 + y = 4$  and  $x^4 - y = 1$

60.  $x^3 - y = 0$  and  $3x^2 - y = 4$

61.  $x + 4y^2 = 4$  and  $x + y^4 = 1$ , for  $x \geq 0$

62.  $x + y^2 = 3$  and  $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 63–70.

63.  $y = 2 \sin x$  and  $y = \sin 2x$ ,  $0 \leq x \leq \pi$

64.  $y = 8 \cos x$  and  $y = \sec^2 x$ ,  $-\pi/3 \leq x \leq \pi/3$

65.  $y = \cos(\pi x/2)$  and  $y = 1 - x^2$

66.  $y = \sin(\pi x/2)$  and  $y = x$

67.  $y = \sec^2 x$ ,  $y = \tan^2 x$ ,  $x = -\pi/4$ , and  $x = \pi/4$

68.  $x = \tan^2 y$  and  $x = -\tan^2 y$ ,  $-\pi/4 \leq y \leq \pi/4$

69.  $x = 3 \sin y \sqrt{\cos y}$  and  $x = 0$ ,  $0 \leq y \leq \pi/2$

70.  $y = \sec^2(\pi x/3)$  and  $y = x^{1/3}$ ,  $-1 \leq x \leq 1$

71. Find the area of the propeller-shaped region enclosed by the curve  $x - y^3 = 0$  and the line  $x - y = 0$ .

72. Find the area of the propeller-shaped region enclosed by the curves  $x - y^{1/3} = 0$  and  $x - y^{1/5} = 0$ .

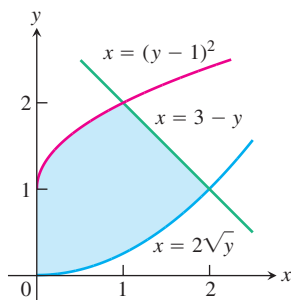
73. Find the area of the region in the first quadrant bounded by the line  $y = x$ , the line  $x = 2$ , the curve  $y = 1/x^2$ , and the  $x$ -axis.

74. Find the area of the “triangular” region in the first quadrant bounded on the left by the  $y$ -axis and on the right by the curves  $y = \sin x$  and  $y = \cos x$ .

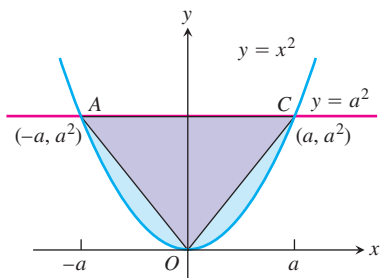
75. The region bounded below by the parabola  $y = x^2$  and above by the line  $y = 4$  is to be partitioned into two subsections of equal area by cutting across it with the horizontal line  $y = c$ .

- a. Sketch the region and draw a line  $y = c$  across it that looks about right. In terms of  $c$ , what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

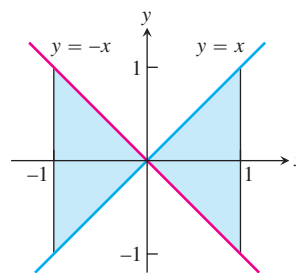
- b. Find  $c$  by integrating with respect to  $y$ . (This puts  $c$  in the limits of integration.)
- c. Find  $c$  by integrating with respect to  $x$ . (This puts  $c$  into the integrand as well.)
76. Find the area of the region between the curve  $y = 3 - x^2$  and the line  $y = -1$  by integrating with respect to **a.**  $x$ , **b.**  $y$ .
77. Find the area of the region in the first quadrant bounded on the left by the  $y$ -axis, below by the line  $y = x/4$ , above left by the curve  $y = 1 + \sqrt{x}$ , and above right by the curve  $y = 2/\sqrt{x}$ .
78. Find the area of the region in the first quadrant bounded on the left by the  $y$ -axis, below by the curve  $x = 2\sqrt{y}$ , above left by the curve  $x = (y - 1)^2$ , and above right by the line  $x = 3 - y$ .



79. The figure here shows triangle  $AOC$  inscribed in the region cut from the parabola  $y = x^2$  by the line  $y = a^2$ . Find the limit of the ratio of the area of the triangle to the area of the parabolic region as  $a$  approaches zero.



80. Suppose the area of the region between the graph of a positive continuous function  $f$  and the  $x$ -axis from  $x = a$  to  $x = b$  is 4 square units. Find the area between the curves  $y = f(x)$  and  $y = 2f(x)$  from  $x = a$  to  $x = b$ .
81. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.
- a.  $\int_{-1}^1 (x - (-x)) dx = \int_{-1}^1 2x dx$
- b.  $\int_{-1}^1 (-x - (x)) dx = \int_{-1}^1 -2x dx$



82. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions  $y = f(x)$  and  $y = g(x)$  and the vertical lines  $x = a$  and  $x = b$  ( $a < b$ ) is

$$\int_a^b [f(x) - g(x)] dx.$$

Give reasons for your answer.

### Theory and Examples

83. Suppose that  $F(x)$  is an antiderivative of  $f(x) = (\sin x)/x$ ,  $x > 0$ . Express

$$\int_1^3 \frac{\sin 2x}{x} dx$$

in terms of  $F$ .

84. Show that if  $f$  is continuous, then

$$\int_0^1 f(x) dx = \int_0^1 f(1 - x) dx.$$

85. Suppose that

$$\int_0^1 f(x) dx = 3.$$

Find

$$\int_{-1}^0 f(x) dx$$

if **a.**  $f$  is odd, **b.**  $f$  is even.

86. **a.** Show that if  $f$  is odd on  $[-a, a]$ , then

$$\int_{-a}^a f(x) dx = 0.$$

**b.** Test the result in part (a) with  $f(x) = \sin x$  and  $a = \pi/2$ .

87. If  $f$  is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x) dx}{f(x) + f(a - x)}$$

by making the substitution  $u = a - x$  and adding the resulting integral to  $I$ .