Section 6.2

1. Write the function as

$$\frac{3}{s^2+4} = \frac{3}{2}\,\frac{2}{s^2+4}\,.$$

Hence $\mathcal{L}^{-1}[Y(s)] = \frac{3}{2}\,sin\,2t\,.$

3. Using *partial fractions*,

$$\frac{2}{s^2+3s-4} = \frac{2}{5}\left[\frac{1}{s-1} - \frac{1}{s+4}\right].$$

Hence $\mathcal{L}^{-1}[Y(s)] = \frac{2}{5}\left(e^t - e^{-4t}\right)$.

5. Note that the denominator s^2+2s+5 is *irreducible* over the reals. Completing the square, $s^2+2s+5 = (s+1)^2+4$. Now convert the function to a *rational function* of the variable $\xi = s+1$. That is,

$$\frac{2s+2}{s^2+2s+5} = \frac{2(s+1)}{(s+1)^2+4}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+4}\right] = 2\,cos\,2t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right] = 2e^{-t}cos\,2t\,.$$

6. Using *partial fractions*,

$$\frac{2s-3}{s^2-4} = \frac{1}{4}\left[\frac{1}{s-2} + \frac{7}{s+2}\right].$$

Hence $\mathcal{L}^{-1}[Y(s)] = \frac{1}{4}(e^{2t}+7e^{-2t})$. Note that we can also write

$$\frac{2s-3}{s^2-4} = 2\,\frac{s}{s^2-4} - \frac{3}{2}\,\frac{2}{s^2-4}.$$

8. Using *partial fractions*,

$$\frac{8s^2-4s+12}{s(s^2+4)} = 3\,\frac{1}{s} + 5\,\frac{s}{s^2+4} - 2\,\frac{2}{s^2+4}.$$

Hence $\mathcal{L}^{-1}[Y(s)] = 3 + 5\,cos\,2t - 2\,sin\,2t$.

9. The denominator $s^2 + 4s + 5$ is *irreducible* over the reals. Completing the square, $s^2 + 4s + 5 = (s+2)^2 + 1$. Now convert the function to a *rational function* of the variable $\xi = s + 2$. That is,

$$\frac{1-2s}{s^2+4s+5} = \frac{5-2(s+2)}{(s+2)^2+1}.$$

We find that

$$\mathcal{L}^{-1}\left[\frac{5}{\xi^2+1} - \frac{2\,\xi}{\xi^2+1}\right] = 5\,sin\,t - 2\,cos\,t.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{1-2s}{s^2+4s+5}\right] = e^{-2t}(5\,sin\,t - 2\,cos\,t).$$

10. Note that the denominator $s^2 + 2s + 10$ is *irreducible* over the reals. Completing the square, $s^2 + 2s + 10 = (s+1)^2 + 9$. Now convert the function to a *rational function* of the variable $\xi = s + 1$. That is,

$$\frac{2s-3}{s^2+2s+10} = \frac{2(s+1)-5}{(s+1)^2+9}.$$

We find that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+9} - \frac{5}{\xi^2+9}\right] = 2\,cos\,3t - \frac{5}{3}\,sin\,3t.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2+2s+10}\right] = e^{-t}\left(2\,cos\,3t - \frac{5}{3}\,sin\,3t\right).$$

12. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - y(0)] + 2\,Y(s) = 0.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) - s - 3 = 0.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s+3}{s^2+3s+2}.$$

Using *partial fractions*,

$$\frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}.$$

Hence $y(t) = \mathcal{L}^{-1}[Y(s)] = 2\,e^{-t} - e^{-2t}$.

13. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] + 2\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - 1 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2-2s+2}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s-1$. That is,

$$\frac{1}{s^2-2s+2} = \frac{1}{(s-1)^2+1}.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{1}{\xi^2+1}\right] = sin\,t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-2s+2}\right] = e^t sin\,t\,.$$

Hence $y(t) = e^t sin\,t$.

15. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] - 2\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) - 2\,Y(s) - 2s + 4 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{2s-4}{s^2-2s-2}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s-1$. Completing the square,

$$\frac{2s-4}{s^2-2s-2} = \frac{2(s-1)-2}{(s-1)^2-3}.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2-3} - \frac{2}{\xi^2-3}\right] = 2\,cosh\,\sqrt{3}\,t - \frac{2}{\sqrt{3}}\,sinh\,\sqrt{3}\,t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s-4}{s^2-2s-2}\right] = e^t\left(2\,cosh\,\sqrt{3}\,t - \frac{2}{\sqrt{3}}\,sinh\,\sqrt{3}\,t\right).$$

16. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2[s\,Y(s) - y(0)] + 5\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 2s\,Y(s) + 5\,Y(s) - 2s - 3 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{2s+3}{s^2+2s+5}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s+1$. That is,

$$\frac{2s+3}{s^2+2s+5} = \frac{2(s+1)+1}{(s+1)^2+4}.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+4} + \frac{1}{\xi^2+4}\right] = 2\,cos\,2t + \frac{1}{2}\,sin\,2t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s+3}{s^2+2s+5}\right] = e^{-t}\left(2\,cos\,2t + \frac{1}{2}\,sin\,2t\right).$$

17. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - 4\left[s^3Y(s) - s^2\,y(0) - s\,y'(0) - y''(0)\right] +$$
$$+ 6\left[s^2\,Y(s) - s\,y(0) - y'(0)\right] - 4[s\,Y(s) - y(0)] + Y(s) = 0$$

Applying the *initial conditions*,

$$s^4Y(s) - 4s^3Y(s) + 6s^2\,Y(s) - 4s\,Y(s) + \,Y(s) - s^2 + 4s - 7 = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2\, - 4s\, + 1} = \frac{s^2 - 4s + 7}{(s-1)^4}\,.$$

Using *partial fractions*,

$$\frac{s^2 - 4s + 7}{(s-1)^4} = \frac{4}{(s-1)^4} - \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2}.$$

Note that $\mathcal{L}[\,t^n\,] = (n!)/s^{n+1}$ and $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$. Hence the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2 - 4s + 7}{(s-1)^4}\right] = \frac{2}{3}\,t^3e^t - t^2e^t + t\,e^t.$$

18. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - \,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^4Y(s) - \,Y(s) - s^3 - s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 - 1}\,.$$

By inspection, it follows that $y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2-1}\right] = cosh\,t\,.$

19. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - 4\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^4Y(s) - \,4Y(s) - s^3 + 2s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 + 2}\,.$$

It follows that $y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+2}\right] = cos\,\sqrt{2}\,t\,.$

20. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \omega^2\,Y(s) = \frac{s}{s^2+4}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + \omega^2\,Y(s) - s = \frac{s}{s^2+4}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2+\omega^2)(s^2+4)} + \frac{s}{s^2+\omega^2}\,.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2+\omega^2)(s^2+4)} = \frac{1}{4-\omega^2}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+4}\right].$$

First note that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = cos\,\omega t \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = cos\,2t\,.$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{4-\omega^2}\,cos\,\omega t - \frac{1}{4-\omega^2}\,cos\,2t + cos\,\omega t \\ &= \frac{5-\omega^2}{4-\omega^2}\,cos\,\omega t - \frac{1}{4-\omega^2}\,cos\,2t\,. \end{aligned}$$

21. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] + 2\,Y(s) = \frac{s}{s^2+1}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - s + 2 = \frac{s}{s^2+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2-2s+2)(s^2+1)} + \frac{s-2}{s^2-2s+2}\,.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2-2s+2)(s^2+1)} = \frac{1}{5}\left[\frac{s-2}{s^2+1} - \frac{s-4}{s^2-2s+2}\right].$$

Thus we can write

$$Y(s) = \frac{1}{5}\,\frac{s}{s^2+1} - \frac{2}{5}\,\frac{1}{s^2+1} + \frac{2}{5}\,\frac{2s-3}{s^2-2s+2}\,.$$

For the *last term*, we note that $s^2 - 2s + 2 = (s-1)^2 + 1$. So that

$$\frac{2s-3}{s^2-2s+2} = \frac{2(s-1)-1}{(s-1)^2+1}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+1} - \frac{1}{\xi^2+1}\right] = 2\,cos\,t - sin\,t\,.$$

Based on the *translation property* of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-2s+2}\right] = e^t(2\,cos\,t - sin\,t).$$

Combining the above, the solution of the IVP is

$$y(t) = \frac{1}{5}\,cos\,t - \frac{2}{5}\,sin\,t + \frac{2}{5}\,e^t(2\,cos\,t - sin\,t)\,.$$

23. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2[s\,Y(s) - y(0)] + Y(s) = \frac{4}{s+1}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 2s\,Y(s) + Y(s) - 2s - 3 = \frac{4}{s+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}\,.$$

First write

$$\frac{2s+3}{(s+1)^2} = \frac{2(s+1)+1}{(s+1)^2} = \frac{2}{s+1} + \frac{1}{(s+1)^2}\,.$$

We note that

$$\mathcal{L}^{-1}\left[\frac{4}{\xi^3} + \frac{2}{\xi} + \frac{1}{\xi^2}\right] = 2\,t^2 + 2 + t\,.$$

So based on the *translation property* of the Laplace transform, the solution of the IVP is