2.3 Exact Equations and Integrating Factors

For first order equations there are a number of integration methods that are applicable to various classes of problems. The most important of these are linear equations and separable equations, which we have discussed previously. Here, we consider a class of equations known as exact equations for which there is also a well-defined method of solution. Keep in mind, however, that those first order equations that can be solved by elementary integration methods are rather special; most first order equations cannot be solved in this way.

Let the differential equation,

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad (1)$$

be given. Suppose that we can identify a function ψ such that,

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y) \quad , \quad \frac{\partial \psi}{\partial y}(x,y) = N(x,y) \quad (2)$$

and such that $\psi(x, y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x. Then

$$M(x,y) + N(x,y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi[x,\phi(x)]$$

and the differential equation (1) becomes

$$\frac{d}{dx}\psi[x,\phi(x)] = 0 \qquad (3)$$

In this case Eq. (1) is said to be an **exact** differential equation. Solutions of Eq. (1), or the equivalent Eq. (3), are given implicitly by

$$\psi(x,y) = c$$

where c is an arbitrary constant.

Theorem 1 : Let the functions M, N, M_y , and N_x , where subscripts denote partial derivatives, be continuous in the rectangular region $R{:}\alpha < x < \beta, \gamma < y < \delta$. Then Eq. (1),

$$M(x,y) + N(x,y)y' = 0$$

is an exact differential equation in R if and only if,

$$M_y(x,y) = N_x(x,y) \qquad (4)$$

at each point of R. That is, there exists a function ψ satisfying Eqs. (2),

$$\psi_x(x,y) = M(x,y) \quad , \quad \psi_y(x,y) = N(x,y)$$

if and only if M and N satisfy Eq. (4).

Solution of Exact Differential Equation

We seek a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x}(x, y) = M(x, y) \qquad (1a)$$

and

$$\frac{\partial \psi}{\partial y}(x, y) = N(x, y) \qquad (1b)$$

Let us assume first that $\psi(x, y)$ satisfies (1a). Then,

$$\psi(x, y) = \int M(x, y)\partial x + h(y) \qquad (2)$$

where $\int M(x, y)\partial x$ indicates a partial integration with respect to *x* holding *y* constant, and *h* is an arbitrary function of *y* only. Differentiating (2) partially with respect to *y*, we obtain

$$\frac{\partial \psi}{\partial y}(x, y) = \frac{\partial}{\partial y}\int M(x, y)\partial x + \frac{d}{dy}h(y)$$

Now, if (1b) is to be satisfied we must have

$$N(x, y) = \frac{\partial}{\partial y}\int M(x, y)\partial x + \frac{d}{dy}h(y) \quad (3)$$

and hence

$$\frac{d}{dy}h(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\partial x$$

since *h* is a function of *y* only, the derivative must also be independent of *x*. That is, in order for (3) to hold

$$N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\partial x$$

must be independent of *x*. That is,

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\partial x\right] = 0$$

Thus we may write

$$h(y) = \int\left[N(x, y) - \int \frac{\partial}{\partial y}M(x, y)\partial x\right]\partial y$$

and

$$\psi(x, y) = \int M(x, y)\partial x + \int\left[N(x, y) - \int \frac{\partial}{\partial y}M(x, y)\partial x\right]\partial y$$

Example : Solve

$$\left(3x^2 + 4xy\right)dx + \left(2x^2 + 2y\right)dy = 0$$

We have,

$$M(x,y) = 3x^2 + 4xy \quad \text{and} \quad N(x,y) = 2x^2 + 2y$$

$$\frac{\partial}{\partial y} M(x,y) = 4x = \frac{\partial}{\partial x} N(x,y)$$

Hence, the equation is exact. Thus we must find find ψ such that

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y) = 3x^2 + 4xy$$

and

$$\frac{\partial \psi}{\partial y}(x,y) = N(x,y) = 2x^2 + 2y$$

From the first of these,

$$\psi(x,y) = \int \left(3x^2 + 4xy\right)\partial x + h(y)$$
$$= x^3 + 2x^2 y + h(y)$$

Then,

$$\frac{\partial \psi}{\partial y}(x,y) = 2x^2 + \frac{d}{dy} h(y)$$

But, we must have

$$\frac{\partial \psi}{\partial y}(x,y) = 2x^2 + 2y$$

Thus,

$$\frac{d}{dy} h(y) = 2y$$

By simple integration we get,

$$h(y) = y^2 + c_0$$

where c_0 is an arbitrary constant, and so

$$\psi(x,y) = x^3 + 2x^2 y + y^2 + c_0$$

Hence, a one parameter family of solution is

$$\psi(x,y) = c_1$$

or

$$x^3 + 2x^2 y + y^2 = c$$

Integrating Factors

It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor. Recall that this is the procedure that we used in solving linear equations. To investigate the possibility of implementing this idea more generally, let us multiply the equation

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1)$$

by a function μ and then try to choose μ so that the resulting equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \qquad (2)$$

is exact. By Theorem 1 Eq. (2) is exact if and only if

$$\frac{\partial}{\partial y}\left[\mu(x,y)M(x,y)\right] = \frac{\partial}{\partial x}\left[\mu(x,y)N(x,y)\right] \qquad (3)$$

Since M and N are given functions, Eq. (3) states that the integrating factor μ must satisfy the first order partial differential equation

$$M(x,y)\frac{\partial}{\partial y}\mu(x,y) - N(x,y)\frac{\partial}{\partial x}\mu(x,y) + \mu(x,y)\left(\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)\right) = 0 \qquad (4)$$

If a function μ satisfying Eq. (4) can be found, then Eq. (2) will be exact. The solution of Eq. (2) can then be obtained by the method described in the first part of this section. The solution found in this way also satisfies Eq. (1), since the integrating factor μ can be canceled out of Eq. (2).

Unfortunately, Eq. (4), which determines the integrating factor μ, is ordinarily at least as difficult to solve as the original equation (1). Therefore, while in principle integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations in which simple *integrating factors can be found occur when* μ is a function of only one of the variables x or y, instead of both. Let us determine necessary conditions on M and N so that Eq. (1) has an integrating factor μ that depends on x only. Assuming that μ is a function of x only, we have

$$\left[\mu(x)M(x,y)\right]_y = \mu(x)M_y(x,y) \quad \text{and} \quad \left[\mu(x)N(x,y)\right]_x = \mu(x)N_x(x,y) + N(x,y)\frac{d\mu}{dx}$$

Thus, if $(\mu M)_y$ is to equal $(\mu N)_x$, it is necessary that

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \qquad (5)$$

If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor μ that also depends only on x; further, $\mu(x)$ can be found by solving Eq. (5), which is both linear and separable.