2.1 Linear Equations with Variable Coefficients

If the function *f* in Eq. (1) depends linearly on the dependent variable *y*, then Eq. (1) is called a first order linear equation. We will usually write the general **first order linear equation** in the form

$$\frac{dy}{dt} + p(t)y = g(t) \qquad (2)$$

where *p* and *g* are given functions of the independent variable *t*.

We will use a method of solution for this type of equations, The method is due to Leibniz; it involves multiplying the differential equation (2) by a certain function μ(t), chosen so that the resulting equation is readily integrable. The function μ(t) is called an **integrating factor** and the main difficulty is to determine how to find it.

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \qquad (3)$$

The question now is whether we can choose $\mu(t)$ so that the left side of Eq. (3) is recognizable as the derivative of some particular expression. If so, then we can integrate Eq. (3), even though we do not know the function *y*. To guide our choice of the integrating factor $\mu(t)$, observe that the left side of Eq. (3) contains two terms and that the first term is part of the result of differentiating the product $\mu(t)y$. Thus, let us try to determine $\mu(t)$ so that the left side of Eq. (3) becomes the derivative of the expression $\mu(t)y$. If we compare the left side of Eq. (3) with the differentiation formula

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y \qquad (4)$$

we note that the first terms are identical and that the second terms also agree, provided we choose $\mu(t)$ to satisfy

$$\frac{d\mu(t)}{dt} = p(t)\mu(t) \qquad (5)$$

If we assume temporarily that $\mu(t)$ is positive, then we have

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t)$$

and consequently,

$$\ln \mu(t) = \int p(t)dt + k$$

By choosing the arbitrary constant *k* to be zero, we obtain the simplest possible function for μ, namely,

$$\mu(t) = \exp\int p(t)dt \qquad (6)$$

Note that $\mu(t)$ is positive for all *t*, as we assumed. Returning to Eq. (3), we have

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)g(t) \qquad (7)$$

integrating both sides,

$$\mu(t)y = \int \mu(s)g(s)ds + c$$

so the general solution of Eq. (2) is

$$y = \frac{\int \mu(s)g(s)ds + c}{\mu(t)} \qquad (8)$$

Observe that, to find the solution given by Eq. (8), two integrations are required: one to obtain $\mu(t)$ from Eq. (6) and the other to obtain y from Eq. (8).

Example : Solve the initial value problem

$$t\,y' + 2y = 4t^2, \qquad (9a)$$
$$y(1) = 2. \qquad (9b)$$

Rewriting Eq. (9a) in the standard form (3), we have,

$$y' + (2/t)y = 4t \qquad (10)$$

so $p(t) = 2/t$ and $g(t) = 4t$. To solve Eq. (10) we first compute the integrating factor $\mu(t)$:

$$\mu(t) = \exp\int (2/t)dt = \exp(2\ln|t|) = t^2.$$

On multiplying Eq. (10) by $\mu(t) = t^2$, we obtain

$$t^2 y' + 2ty = 4t^3,$$

and therefore

$$t^2 y = t^4 + c,$$

where c is an arbitrary constant. It follows that

$$y = t^2 + c/t^2,$$

is the general solution of Eq. (9a). To satisfy the initial condition (9b) it is necessary to choose $c = 1$; thus

$$y = t^2 + 1/t^2,$$

is the solution of the initial value problem (9a,b).