CHAPTER 3

Classifications of First-Order Differential Equations

STANDARD FORM AND DIFFERENTIAL FORM

Standard form for a first-order differential equation in the unknown function $y(x)$ is

$$y' = f(x, y) \tag{3.1}$$

where the derivative y' appears only on the left side of (*3.1*). Many, but not all, first-order differential equations can be written in standard form by algebraically solving for y' and then setting $f(x, y)$ equal to the right side of the resulting equation.

The right side of (*3.1*) can always be written as a quotient of two other functions $M(x, y)$ and $-N(x, y)$. Then (*3.1*) becomes $dy/dx = M(x, y)/-N(x, y)$, which is equivalent to the *differential form*

$$M(x, y)dx + N(x, y)dy = 0 \tag{3.2}$$

LINEAR EQUATIONS

Consider a differential equation in standard form (*3.1*). If $f(x, y)$ can be written as $f(x, y) = -p(x)y + q(x)$ (that is, as a function of x times y, plus another function of x), the differential equation is *linear*. First-order linear differential equations can always be expressed as

$$y' + p(x)y = q(x) \tag{3.3}$$

Linear equations are solved in Chapter 6.

BERNOULLI EQUATIONS

A *Bernoulli* differential equation is an equation of the form

$$y' + p(x)y = q(x)y^n \tag{3.4}$$

where n denotes a real number. When $n = 1$ or $n = 0$, a Bernoulli equation reduces to a linear equation. Bernoulli equations are solved in Chapter 6.

HOMOGENEOUS EQUATIONS

A differential equation in standard form (*3.1*) is *homogeneous* if

$$f(tx, ty) = f(x, y) \tag{3.5}$$

for every real number t. Homogeneous equations are solved in Chapter 4.

Note: In the general framework of differential equations, the word "homogeneous" has an entirely different meaning (see Chapter 8). Only in the context of first-order differential equations does "homogeneous" have the meaning defined above.

SEPARABLE EQUATIONS

Consider a differential equation in differential form (*3.2*). If $M(x, y) = A(x)$ (a function only of x) and $N(x, y) = B(y)$ (a function only of y), the differential equation is *separable*, or has its *variables separated.* Separable equations are solved in Chapter 4.

EXACT EQUATIONS

A differential equation in differential form (*3.2*) is *exact* if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \tag{3.6}$$

Exact equations are solved in Chapter 5 (where a more precise definition of exactness is given).

Solved Problems

3.1. Write the differential equation $xy' - y^2 = 0$ in standard form.

Solving for y', we obtain $y' = y^2/x$ which has form (*3.1*) with $f(x, y) = y^2/x$.

3.2. Write the differential equation $e^x y' + e^{2x} y = \sin x$ in standard form.

Solving for y', we obtain

$$e^x y' = -e^{2x} y + \sin x$$

or

$$y' = -e^x y + e^{-x} \sin x$$

which has form (*3.1*) with $f(x, y) = -e^x y + e^{-x} \sin x$.

3.3. Write the differential equation $(y' + y)^5 = \sin (y'/x)$ in standard form.

This equation cannot be solved algebraically for y', and *cannot* be written in standard form.

3.4. Write the differential equation $y(yy' - 1) = x$ in differential form.

Solving for y', we have

$$y^2 y' - y = x$$
$$y^2 y' = x + y$$

or
$$y' = \frac{x+y}{y^2} \tag{1}$$

which is in standard form with $f(x, y) = (x + y)/y^2$. There are infinitely many different differential forms associated with (*1*). Four such forms are:

(*a*) Take $M(x, y) = x + y$, $N(x, y) = -y^2$. Then

$$\frac{M(x,y)}{-N(x,y)} = \frac{x+y}{-(-y^2)} = \frac{x+y}{y^2}$$

and (*1*) is equivalent to the differential form

$$(x + y)dx + (-y^2)dy = 0$$

(*b*) Take $M(x, y) = -1$, $N(x,y) = \dfrac{y^2}{x+y}$. Then

$$\frac{M(x,y)}{-N(x,y)} = \frac{-1}{-y^2/(x+y)} = \frac{x+y}{y^2}$$

and (*1*) is equivalent to the differential form

$$(-1)dx + \left(\frac{y^2}{x+y}\right)dy = 0$$

(*c*) Take $M(x,y) = \dfrac{x+y}{2}$, $N(x,y) = \dfrac{-y^2}{2}$. Then

$$\frac{M(x,y)}{-N(x,y)} = \frac{(x+y)/2}{-(-y^2/2)} = \frac{x+y}{y^2}$$

and (*1*) is equivalent to the differential form

$$\left(\frac{x+y}{2}\right)dx + \left(\frac{-y^2}{2}\right)dy = 0$$

(*d*) Take $M(x,y) = \dfrac{-x-y}{x^2}$, $N(x,y) = \dfrac{y^2}{x^2}$. Then

$$\frac{M(x,y)}{-N(x,y)} = \frac{(-x-y)/x^2}{-y^2/x^2} = \frac{x+y}{y^2}$$

and (*1*) is equivalent to the differential form

$$\left(\frac{-x-y}{x^2}\right)dx + \left(\frac{y^2}{x^2}\right)dy = 0$$

3.5. Write the differential equation $dy/dx = y/x$ in differential form

This equation has infinitely many differential forms. One is

$$dy = \frac{y}{x}dx$$

which can be written in form (*3.2*) as

$$\frac{y}{x}dx + (-1)dy = 0 \tag{1}$$

Multiplying (*1*) through by x, we obtain

$$y\,dx + (-x)dy = 0 \tag{2}$$

as a second differential form. Multiplying (*1*) through by $1/y$, we obtain

$$\frac{1}{x}dx + \frac{-1}{y}dy = 0 \tag{3}$$

as a third differential form. Still other differential forms are derived from (*1*) by multiplying that equation through by any other function of x and y.

3.6. Write the differential equation $(xy + 3)dx + (2x - y^2 + 1)dy = 0$ in standard form.

This equation is in differential form. We rewrite it as

$$(2x - y^2 + 1)\,dy = -(xy + 3)\,dx$$

which has the standard form

$$\frac{dy}{dx} = \frac{-(xy+3)}{2x - y^2 + 1}$$

or

$$y' = \frac{xy+3}{y^2 - 2x - 1}$$

3.7. Determine if the following differential equations are linear:

(*a*) $y' = (\sin x)y + e^x$ (*b*) $y' = x \sin y + e^x$ (*c*) $y' = 5$ (*d*) $y' = y^2 + x$

(*e*) $y' + xy^5 = 0$ (*f*) $xy' + y = \sqrt{y}$ (*g*) $y' + xy = e^x y$ (*h*) $y' + \frac{x}{y} = 0$

(*a*) The equation is linear; here $p(x) = -\sin x$ and $q(x) = e^x$.
(*b*) The equation is not linear because of the term $\sin y$.
(*c*) The equation is linear; here $p(x) = 0$ and $q(x) = 5$.
(*d*) The equation is not linear because of the term y^2.
(*e*) The equation is not linear because of the y^5 term.
(*f*) The equation is not linear because of the $y^{1/2}$ term.
(*g*) The equation is linear. Rewrite it as $y' + (x - e^x)y = 0$ with $p(x) = x - e^x$ and $q(x) = 0$.
(*h*) The equation is not linear because of the $1/y$ term.

3.8. Determine whether any of the differential equations in Problem 3.7 are Bernoulli equations.

All of the linear equations are Bernoulli equations with $n = 0$. In addition, three of the nonlinear equations, (*e*), (*f*) and (*h*), are as well. Rewrite (*e*) as $y' = -xy^5$; it has form (*3.4*) with $p(x) = 0$, $q(x) = -x$, and $n = 5$. Rewrite (*f*) as

$$y' + \frac{1}{x}y = \frac{1}{x}y^{1/2}$$

It has form (*3.4*) with $p(x) = q(x) = 1/x$ and $n = 1/2$. Rewrite (*h*) as $y' = -xy^{-1}$ with $p(x) = 0$, $q(x) = -x$, and $n = -1$.

3.9. Determine if the following differential equations are homogeneous:

(*a*) $y' = \dfrac{y+x}{x}$ (*b*) $y' = \dfrac{y^2}{x}$ (*c*) $y' = \dfrac{2xye^{x/y}}{x^2 + y^2 \sin\frac{x}{y}}$ (*d*) $y' = \dfrac{x^2 + y}{x^3}$

(*a*) The equation is homogeneous, since

$$f(tx, ty) = \frac{ty + tx}{tx} = \frac{t(y+x)}{tx} = \frac{y+x}{x} = f(x, y)$$

(*b*) The equation is not homogeneous, since

$$f(tx, ty) = \frac{(ty)^2}{tx} = \frac{t^2 y^2}{tx} = t\frac{y^2}{x} \neq f(x, y)$$

(*c*) The equation is homogeneous, since

$$f(tx, ty) = \frac{2(tx)(ty)e^{tx/ty}}{(tx)^2 + (ty)^2 \sin\dfrac{tx}{ty}} = \frac{t^2 2xye^{x/y}}{t^2x^2 + t^2y^2 \sin\dfrac{x}{y}}$$

$$= \frac{2xye^{x/y}}{x^2 + y^2 \sin\dfrac{x}{y}} = f(x, y)$$

(*d*) The equation is not homogeneous, since

$$f(tx, ty) = \frac{(tx)^2 + ty}{(tx)^3} = \frac{t^2x^2 + ty}{t^3x^3} = \frac{tx^2 + y}{t^2x^3} \neq f(x, y)$$

3.10. Determine if the following differential equations are separable:

(*a*) $\sin x\, dx + y^2 dy = 0$ (*b*) $xy^2 dx - x^2y^2 dy = 0$ (*c*) $(1 + xy)dx + y\, dy = 0$

(*a*) The differential equation is separable; here $M(x, y) = A(x) = \sin x$ and $N(x, y) = B(y) = y^2$.

(*b*) The equation is not separable in its present form, since $M(x, y) = xy^2$ is not a function of x alone. But if we divide both sides of the equation by x^2y^2, we obtain the equation $(1/x)dx + (-1)dy = 0$, which is separable. Here, $A(x) = 1/x$ and $B(y) = -1$.

(*c*) The equation is not separable, since $M(x, y) = 1 + xy$, which is not a function of x alone.

3.11. Determine whether the following differential equations are exact:

(*a*) $3x^2y\, dx + (y + x^3)dy = 0$ (*b*) $xy\, dx + y^2 dy = 0$

(*a*) The equation is exact; here $M(x, y) = 3x^2y$, $N(x, y) = y + x^3$, and $\partial M/\partial y = \partial N/\partial x = 3x^2$.

(*b*) The equation is not exact. Here $M(x, y) = xy$ and $N(x, y) = y^2$; hence $\partial M/\partial y = x$, $\partial N/\partial x = 0$, and $\partial M/\partial y \neq \partial N/\partial x$.

3.12. Determine whether the differential equation $y' = y/x$ is exact.

Exactness is only defined for equations in differential form, not standard form. The given differential equation has many differential forms. One such form is given in Problem 3.5, Eq. (*1*), as

$$\frac{y}{x}dx + (-1)dy = 0$$

Here $M(x, y) = y/x$, $N(x, y) = -1$,

$$\frac{\partial M}{\partial y} = \frac{1}{x} \neq 0 = \frac{\partial N}{\partial x}$$

and the equation is not exact. A second differential form for the same differential equation is given in Eq. (*3*) of Problem 3.5 as

$$\frac{1}{x}dx + \frac{-1}{y}dy = 0$$

Here $M(x, y) = 1/x$, $N(x, y) = -1/y$,

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

and the equation is exact. Thus, a given differential equation has many differential forms, some of which may be exact.

3.13. Prove that a separable equation is always exact.

For a separable differential equation, $M(x, y) = A(x)$ and $N(x, y) = B(y)$. Thus,

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial A(x)}{\partial y} = 0 \quad \text{and} \quad \frac{\partial N(x,y)}{\partial x} = \frac{\partial B(y)}{\partial x} = 0$$

Since $\partial M/\partial y = \partial N/\partial x$, the differential equation is exact.

3.14. A theorem of first-order differential equations states that if $f(x, y)$ and $\partial f(x, y)/\partial y$ are continuous in a rectangle $\mathcal{R}$: $|x - x_0| \le a$, $|y - y_0| \le b$, then there exists an interval about x_0 in which the initial-value problem $y' = f(x, y)$; $y(x_0) = y_0$ has a unique solution. The initial-value problem $y' = 2\sqrt{|\,y\,|}$; $y(0) = 0$ has the two solutions $y = x\,|x|$ and $y \equiv 0$. Does this result violate the theorem?

No. Here, $f(x, y) = 2\sqrt{|\,y\,|}$ and, therefore, $\partial f/\partial y$ does not exist at the origin.

Supplementary Problems

In Problems 3.15 through 3.25, write the given differential equations in standard form.

3.15. $xy' + y^2 = 0$

3.16. $e^x y' - x = y'$

3.17. $(y')^3 + y^2 + y = \sin x$

3.18. $xy' + \cos(y' + y) = 1$

3.19. $e^{(y' + y)} = x$

3.20. $(y')^2 - 5y' + 6 = (x + y)(y' - 2)$

3.21. $(x - y)dx + y^2 dy = 0$

3.22. $\dfrac{x + y}{x - y}dx - dy = 0$

3.23. $dx + \dfrac{x + y}{x - y}dy = 0$

3.24. $(e^{2x} - y)dx + e^x dy = 0$

3.25. $dy + dx = 0$

In Problems 3.26 through 3.35, differential equations are given in both standard and differential form. Determine whether the equations in standard form are homogeneous and/or linear, and, if not linear, whether they are Bernoulli; determine whether the equations in differential form, *as given*, are separable and/or exact.

3.26. $y' = xy$; $\quad xydx - dy = 0$

3.27. $y' = xy$; $\quad x\,dx - \dfrac{1}{y}dy = 0$

3.28. $y' = xy + 1$; $\quad (xy + 1)dx - dy = 0$

3.29. $y' = \dfrac{x^2}{y^2}$; $\quad \dfrac{x^2}{y^2}dx - dy = 0$

3.30. $y' = \dfrac{x^2}{y^2}$; $-x^2 dx + y^2 dy = 0$

3.31. $y' = -\dfrac{2y}{x}$; $2xydx + x^2 dy = 0$

3.32. $y' = \dfrac{xy^2}{x^2 y + y^3}$; $xy^2 dx - (x^2 y + y^3) dy = 0$

3.33. $y' = \dfrac{-xy^2}{x^2 y + y^2}$; $xy^2 dx + (x^2 y + y^2) dy = 0$

3.34. $y' = x^3 y + xy^3$; $(x^2 + y^2) dx - \dfrac{1}{xy} dy = 0$

3.35. $y' = 2xy + x$; $(2xye^{-x^2} + xe^{-x^2}) dx - e^{-x^2} dy = 0$