1.1 Introduction

Differential equations occur in connection with numereous problems that are encountered in various branches of science and engineering. In each physical problem the objects involved obey certain laws. These laws involve various rates of change of one or more quantities with respect to other quantities. Such rates of change are expressed mathematically by derivatives In the mathematical formulation of each of physical problems, the various rate of change are thus expressed by various derivatives and the laws governing the physical process become mathematical expressions involving derivatives, that is differential equations.

The aim of this first course is to make the reader familiar with some basic terminology and basic problems pertinent to the differential equations.

In what follows all variables and functions we will consider are assumed to be real-valued unless the contrary is stated explicitly

$$x, y, z; u, v, w \ldots \in \Re$$

Definition 1.1. An equation involving some unknown functions together with their derivative (of finite order) is called a differential equation.

For examples of differential equations we list the following.

$$y = \phi(t), \quad \mathrm{t} \in \Re$$

$$y' = t \qquad (1a)$$

$$(y')^2 = t^2 + 2 \qquad (1b)$$

$$t^2 y'' + 2y = t^3 \qquad (1c)$$

$$t^2 y'' + 2(y')^2 + 3ty = \sin t \qquad (1d)$$

$$y' + e^t y + te^y = 0 \qquad (1e)$$

and

$$z = z(x, y), \quad x, y \in \Re$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + 4z = 0, \quad x^2 + y^2 > 4 \qquad (2a)$$

$$x \frac{\partial^2 z}{\partial x \partial y} = z \sin x + x \sin z, \quad x^2 + y^2 < 1 \qquad (2b)$$

$$z \frac{\partial z}{\partial x} + 3x \frac{\partial z}{\partial y} + 2z^3 = e^x, \quad x, y \in \Re^2 \qquad (2c)$$

From this brief list of differential equations it is clear that various variables and derivatives in a differential equation can occur in a variety of ways. Clearly some kind of classification must be made. To begin with, we classify the differential equation according to whether there is one or more than one independent variable involved.

Definition 1.2. A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

The differential equation in (1.1a-e) are ordinary differential equations of the form

$$F\left[\, t, y(t), y'(t), \ldots, y^{(n)}(t) \,\right] = 0 \qquad (3a)$$

or

$$y^{(n)} = f\left[\, t, y(t), y'(t), \ldots, y^{(n-1)}(t) \,\right] \qquad (3b)$$

These equations are related to a function *y* depending on only one independent variable *t*.

Definition 1.3. A differential equation involving partial derivative of one or more dependent variables with respect to more than one independent variable is called a partial differential equation

(2a-b) are partial differential equations of the form

$$F\left(x, y, z(x,y), z_x, z_y, z_{xx}, z_{xy}, z_{yy}, \ldots\right) = 0 \qquad (4)$$

We further classify differential equations, both ordinary and partial, according to the order of the highest derivative appearing in the equation. For this purpose we give the following definition.

Definition 1.4. The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

The ODE (ordinary differential equation) (1a,b,e) and the PDE (2c) are of the first order, since the highest derivative involved is a .rst derivative. (1c,d) and (2a,b) are of the second order.

Proceeding with our study of ODEs, we now introduce the concept of linearity applied to such equations. This concept will enable us to classify these equations still further.

Definition 1.5. A *linear* ODE of order *n*, in the dependent variable *y* and the independent variable *t*; is an equation that is in, or can be expressed in the form

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \ldots\ldots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = g(t), \quad t \in I \qquad (5)$$

where $a_0(t)$ is not identically zero

OBSERVE THAT

1- The dependent variable *y* and its various derivatives occur to the first degree only.
2- No products of *y* and/or any of its derivatives are present.
3- No transcendental functions of *y* and/or its derivatives occur.

The following ODEs are both linear

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0$$
$$\frac{d^4y}{dt^4}+t^2\frac{d^3y}{dt^3}+\frac{dy}{dt}=te^t$$

Indeed, y and its various derivatives occur to the first degree only and that no products of y and/or its derivatives are present.

A *nonlinear differential equation* is an ODE that is not linear. The following ODEs are all nonlinear:

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y^2=0$$
$$\frac{d^2y}{dt^2}+5\left(\frac{dy}{dt}\right)^3+6y=0$$
$$y''+3\sin y=0$$

1.2 Solutions

Definition 1.6. (Explicit Solution): A relation $y=\phi(t)$, defined for real t in a real interval I is called an explicit solution to

$$\phi^{(n)}(t)=f\left[\,t,\phi(t),\phi'(t),\ldots\ldots,\phi^{(n-1)}(t)\,\right],\qquad t\in I \qquad (6)$$

if all the derivatives $\phi'(t),\ldots\ldots,\phi^{(n)}(t)$ are defined on I and satisfy the equation (6) *identically*.

For the differential equation

$$y''+y=0,\qquad t\in\Re$$

the explicit solution is

$$y\,(t) = A\cos t + B\sin t$$

where A and B are arbitrary constants.

Notice that, sometimes we can not find an explicit expression of the solution. But, we can easily get it implicitely. We refer to it at implicit solution.

Definition 1.7. (Implicit Solution): A relation $f(t,y)=0$ which does not involve any derivatives of y is called an implicit solution of (6) if this relation defines at least one explicit solution of (6) on this interval. Then we may say that a solution of the differential equation (6) is a relation-explicit or implicit-between t and y not containing derivatives, which identically satisfies (6).

The relation

$$t^2 + y^2 - 25 = 0$$

is an implicit solution of the differential equation

$$t + yy' = 0$$

on the interval -5 < t < 5.

Some Important Questions.

In general we do not have solutions of differential equations readily available. Thus a fundamental question is the following: Does an equation of the form (3) always have a solution? The answer is "No." Merely writing down an equation of the form (3) does not necessarily mean that there is a function $y = \phi(t)$ that satisfies it. So, how can we tell whether some particular equation has a solution? This is the question of *existence* of a solution, and it is answered by theorems stating that under certain restrictions on the function F in Eq. (3), the equation always has solutions.

Second, if we assume that a given differential equation has at least one solution, the question arises as to how many solutions it has, and what additional conditions must be specified to single out a particular solution. This is the question of *uniqueness.*

A third important question is: Given a differential equation of the form (3), can we actually determine a solution, and if so, how? Note that if we find a solution of the given equation, we have at the same time answered the question of the existence of a solution. However, without knowledge of existence theory we might, for example, use a computer to find a numerical approximation to a "solution" that does not exist. On the other hand, even though we may know that a solution exists, it may be that the solution is not expressible in terms of the usual elementary functions—polynomial, trigonometric, exponential, logarithmic, and hyperbolic functions. Unfortunately, this is the situation for most differential equations. Thus, while we discuss elementary methods that can be used to obtain solutions of certain relatively simple problems, it is also important to consider methods of a more general nature that can be applied to more difficult problems.

1.3. Initial Value Problem

Definition 1.8. By an initial value problem for an n-th order differential equation

$$\phi^{(n)}(t) = f\left[\, t, \phi(t), \phi'(t), \ldots\ldots, \phi^{(n-1)}(t) \,\right], \qquad t \in I$$

we mean: Find a solution to the differential equation on an interval I, that satisfies at t_0 the n initial conditions

$$\begin{aligned} y(t_0) &= y_0 \\ \frac{d}{dt} y(t_0) &= y_1 \\ &\vdots \\ \frac{d^{n-1}}{dt^{n-1}} y(t_0) &= y_{n-1} \end{aligned}$$

where $t_0 \in I$ and $y_0, y_1, \ldots\ldots, y_{n-1}$ are given constants.

2. First Order Differential Equations

This chapter deals with differential equations of first order,

$$\frac{dy}{dt} = f(t, y) \qquad (1)$$

where f is a given function of two variables. Any differentiable function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a solution, and our object is to determine whether such functions exist and, if so, to develop methods for finding them. Unfortunately, for an arbitrary function f , there is no general method for solving the equation in terms of elementary functions. Instead, we will describe several methods, each of which is applicable to a certain subclass of first order equations. The most important of these are linear equations, separable equations and exact equations.

Existence and Uniqueness Theorem

Theorem 1: If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$\frac{dy}{dt} + p(t)y = g(t) \qquad (1)$$

for each t in I , and that also satisfies the initial condition

$$y(t_0) = y_0 , \qquad (2)$$

where y_0 is an arbitrary prescribed initial value.

Theorem 1 states that the given initial value problem *has* a solution and also that the problem has *only one* solution. In other words, the theorem asserts both the *existence* and *uniqueness* of the solution of the initial value problem (1), (2). In addition, it states that the solution exists throughout any interval I containing the initial point t_0 in which the coefficients p and g are continuous. That is, the solution can be discontinuous or fail to exist only at points where at least one of p and g is discontinuous.

The solution is,

$$y = \frac{\int_{t_0}^{t} \mu(s) g(s) ds + y_0}{\mu(t)}$$

where

$$\mu(t) = \exp \int_{t_0}^{t} p(s) ds$$

Theorem 2: Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$\frac{dy}{dt} = f(t, y) \quad , \quad y(t_0) = y_0 .$$