2.2 Seperable Equations

The general first order equation is

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

Linear equations were considered in the preceding section, but if Eq. (1) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first order equations for which a direct integration process can be used. To identify this class of equations we first rewrite Eq. (1) in the form,

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \qquad (2)$$

In the event that M is a function of x only and N is a function of y only, then Eq. (2) becomes,

$$M(x) + N(y)\frac{dy}{dx} = 0 \qquad (3)$$

Such an equation is said to be **separable**, because if it is written in the differential form,

$$M(x)dx + N(y)dy = 0 \qquad (4)$$

then, if you wish, terms involving each variable may be separated by the equals sign. The differential form (4) is also more symmetric and tends to diminish the distinction between independent and dependent variables.

A seperable equation can be solved by integrating the functions M and N. For example the equation,

$$(x-4)y^4dx - x^3(y^2-3)dy = 0$$

is a seperable equation

$$\frac{(x-4)}{x^3}dx - \frac{(y^2-3)}{y^4}dy = 0$$

integrating we obtain the solution,

$$-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$$

where c is an arbitrary constant.

Essentially the same procedure can be followed for any separable equation. Returning to Eq. (3), let H_1 and H_2 be any antiderivatives of M and N, respectively. Thus,

$$H_1'(x) = M(x) \quad , \quad H_2'(y) = N(y) \qquad (5)$$

then eqn. (3) becomes,

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0, \qquad (6)$$

According to the chain rule,

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y),$$

Eqn. (5) becomes,

$$\frac{d}{dx}\left[H_1(x) + H_2(y)\right] = 0 \qquad (7)$$

By integrating Eq. (7) we obtain,

$$H_1(x) + H_2(y) = c \qquad (8)$$

where c is an arbitrary constant. Any differentiable function $y = \varphi(x)$ that satisfies Eq. (8) is a solution of Eq. (3); in other words, Eq. (8) defines the solution implicitly rather than explicitly. In practice, Eq. (8) is usually obtained from Eq. (4) by integrating the first term with respect to x and the second term with respect to y.

If, in addition to the differential equation, an initial condition

$$y(x_0) = y_0, \qquad (9)$$

is prescribed, then the solution of Eq. (3) satisfying this condition is obtained by setting $x = x_0$ and $y = y_0$ in Eq. (8). This gives,

$$H_1(x_0) + H_2(y_0) = c \qquad (10)$$

Substituting this value of c in Eq. (8) and noting that

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)ds \quad , \quad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)ds$$

We obtain,

$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0 \qquad (11)$$

Equation (11) is an implicit representation of the solution of the differential equation (3) that also satisfies the initial condition (9). You should bear in mind that the determination of an explicit formula for the solution requires that Eq. (11) be solved for y as a function of x. Unfortunately, it is often impossible to do this analytically; in such cases one can resort to numerical methods to find approximate values of y for given values of x.