Bernoulli Differential Equations

A differential equation of the form

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \cdot y^n, \quad n = \text{Constant}, \tag{1}$$

is called a Bernoulli differential equation. Except when $n = 0$ or 1, the equation is nonlinear.

- When $n = 0$, the equation reduces to a linear first-order equation

$$\frac{dy}{dx} + P(x) \cdot y = Q(x).$$

- When $n = 1$, the equation can be written as

$$\frac{dy}{dx} = [Q(x) - P(x)]\, y. \qquad \text{✍ Variable separable}$$

❧ In the following, the case when $n \neq 0, 1$ is considered.

For $n < 0$, $y \neq 0$. When $n > 0$, $y = 0$ is a solution of the differential equation. For $y \neq 0$, dividing both sides of equation (1) by y^n yields

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\cdot y^{1-n} = Q(x).$$

Letting $u = y^{1-n}$, when it is defined, one has

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (1-n)\,y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} \quad\Longrightarrow\quad y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x},$$

and the equation becomes

$$\frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x}+P(x)\,u=Q(x),$$

or

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\underbrace{(1-n)P(x)}_{\bar{P}(x)}\cdot u=\underbrace{(1-n)Q(x)}_{\bar{Q}(x)},\qquad n\neq 1. \tag{2}$$

Hence, a Bernoulli differential equation is transformed to a linear first-order equation (2) in the new variable u.

Bernoulli Differential Equations

1. $$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\cdot y = Q(x)\cdot y^n$$

$$u = y^{1-n} \implies \frac{\mathrm{d}u}{\mathrm{d}x} + \underbrace{(1-n)P(x)}_{\bar{P}(x)}\cdot u = \underbrace{(1-n)Q(x)}_{\bar{Q}(x)},$$ ✍ Linear first-order

2. $$\frac{\mathrm{d}x}{\mathrm{d}y} + P(y)\cdot x = Q(y)\cdot x^n$$

$$u = x^{1-n} \implies \frac{\mathrm{d}u}{\mathrm{d}y} + \underbrace{(1-n)P(y)}_{\bar{P}(y)}\cdot u = \underbrace{(1-n)Q(y)}_{\bar{Q}(y)},$$ ✍ Linear first-order

Example 2.33

Solve $\quad 2xyy' = y^2 - 2x^3, \quad y(1) = 2.$

The differential equation can be written as

$$y' - \frac{1}{2x} \cdot y = -x^2 \cdot \frac{1}{y}, \qquad \text{Bernoulli DE with } n = -1$$

Multiplying both sides of the equation by y yields

$$yy' - \frac{1}{2x} \cdot y^2 = -x^2.$$

Letting $u = y^2 \implies \dfrac{du}{dx} = 2y\dfrac{dy}{dx}$, the equation becomes

$$\frac{1}{2}\frac{du}{dx} - \frac{1}{2x}\cdot u = -x^2 \implies \frac{du}{dx} - \frac{1}{x}\cdot u = -2x^2.$$

The equation is linear first-order of the form

$$\frac{du}{dx} + \bar{P}(x)\cdot u = \bar{Q}(x), \qquad \bar{P}(x) = -\frac{1}{x}, \quad \bar{Q}(x) = -2x^2.$$

The following quantities can be evaluated

$$\int \bar{P}(x)\,dx = \int -\frac{1}{x}\,dx = -\ln|x|, \quad e^{\int \bar{P}(x)\,dx} = e^{-\ln|x|} = \frac{1}{x}, \quad e^{-\int \bar{P}(x)\,dx} = x,$$

$$\int \bar{Q}(x)\,e^{\int \bar{P}(x)\,dx}dx = \int -2x^2\cdot\frac{1}{x}\,dx = -x^2.$$

Hence

$$u = \mathrm{e}^{-\int \bar{P}(x)\,\mathrm{d}x}\left[\int \bar{Q}(x)\,\mathrm{e}^{\int \bar{P}(x)\,\mathrm{d}x}\mathrm{d}x + C\right] = x\,(-x^2 + C),$$

i.e.,

$$y^2 = x\,(-x^2 + C).$$ ✍ General solution

The constant C can be determined from the initial condition $y(1) = 2$

$$2^2 = 1 \cdot (-1^2 + C) \implies C = 5.$$

The particular solution is

$$y^2 = x\,(5 - x^2).$$ ✍ Particular solution

Example 2.34

Solve $3y\,dx - x\left(3x^3 y \ln|y| + 1\right)dy = 0, \quad y \neq 0.$

The differential equation can be written as

$$\frac{dx}{dy} - \frac{1}{3y}\cdot x = \ln|y| \cdot x^4.$$

✍ Bernoulli DE with $n=4$

Dividing both sides of the equation by x^4 yields

$$\frac{1}{x^4}\frac{dx}{dy} - \frac{1}{3y}\cdot\frac{1}{x^3} = \ln|y|.$$

Letting $u = \dfrac{1}{x^3} \implies \dfrac{du}{dy} = -\dfrac{3}{x^4}\dfrac{dx}{dy}$, the equation becomes

$$-\frac{1}{3}\frac{du}{dy} - \frac{1}{3y}\cdot u = \ln|y| \implies \frac{du}{dy} + \frac{1}{y}\cdot u = -3\ln|y|.$$

The equation is linear first-order of the form

$$\frac{\mathrm{d}u}{\mathrm{d}y} + \bar{P}(y)\cdot u = \bar{Q}(y), \qquad \bar{P}(y) = \frac{1}{y}, \quad \bar{Q}(y) = -3\ln|y|.$$

The following quantities can be evaluated

$$\int \bar{P}(y)\,\mathrm{d}y = \int \frac{1}{y}\,\mathrm{d}y = \ln|y|, \quad \mathrm{e}^{\int \bar{P}(y)\,\mathrm{d}y} = \mathrm{e}^{\ln|y|} = y, \quad \mathrm{e}^{-\int \bar{P}(y)\,\mathrm{d}y} = \frac{1}{y},$$

$$\int \bar{Q}(y)\,\mathrm{e}^{\int \bar{P}(y)\,\mathrm{d}y}\mathrm{d}y = \int -3\ln|y|\cdot y\,\mathrm{d}y = -\tfrac{3}{2}\int \ln|y|\,\mathrm{d}(y^2)$$ Integration by parts

$$= -\tfrac{3}{2}\left(y^2\ln|y| - \int y^2\cdot\frac{1}{y}\,\mathrm{d}y\right) = -\tfrac{3}{2}\left(y^2\ln|y| - \tfrac{1}{2}y^2\right).$$

Hence

$$u = \mathrm{e}^{-\int \bar{P}(y)\,\mathrm{d}y}\left[\int \bar{Q}(y)\,\mathrm{e}^{\int \bar{P}(y)\,\mathrm{d}y}\,\mathrm{d}y + C\right] = \frac{1}{y}\left[-\tfrac{3}{4}y^2\left(2\ln|y| - 1\right) + C\right].$$

Replacing u by the original variable results in the general solution

$$\frac{4}{x^3} = -3y\left(2\ln|y| - 1\right) + \frac{C}{y}.$$