Homogeneous Equations

Equations of the type

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{1}$$

are called *homogeneous differential equations*. For example,

$$g(x, y) = \frac{x^2 + 3y^2}{x^2 - xy + y^2} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{1 - \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2} = f\left(\frac{y}{x}\right),$$

$$g(x, y) = \ln x - \ln y = \ln\left(\frac{x}{y}\right) = -\ln\left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right).$$

A homogeneous equation can be converted to a variable separable equation using a transformation of variables. Let $v = \frac{y}{x}$ be the new dependent variable, while x is still the independent variable. Hence

$$y = xv \implies \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substituting into differential equation (1) leads to

$$v + x\frac{dv}{dx} = f(v) \implies x\frac{dv}{dx} = f(v) - v.$$

The transformed differential equation is variable separable. Integrating both sides gives the general solution

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + C.$$

Example 2.4

Solve $$\frac{dy}{dx}+\frac{x}{y}+2=0, \quad y\neq 0, \quad y(0)=1.$$

The differential equation is homogeneous. Letting $v=\frac{y}{x}$,

$$y=xv \implies \frac{dy}{dx}=v+x\frac{dv}{dx},$$

the differential equation becomes

$$v+x\frac{dv}{dx}+\frac{1}{v}+2=0 \implies x\frac{dv}{dx}=-\left(v+\frac{1}{v}+2\right) \implies x\frac{dv}{dx}=-\frac{(v+1)^2}{v}.$$

$v \neq -1$, separating the variables yields

$$\frac{v}{(v+1)^2}\,\mathrm{d}v = -\frac{1}{x}\,\mathrm{d}x.$$

Integrating both sides gives

$$\int \frac{v}{(v+1)^2}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x + C.$$

Since

$$\int \frac{v}{(v+1)^2}\,\mathrm{d}v = \int \frac{(v+1)-1}{(v+1)^2}\,\mathrm{d}v = \int \left\{\frac{1}{v+1} - \frac{1}{(v+1)^2}\right\}\mathrm{d}v$$

$$= \int \frac{1}{v+1}\,\mathrm{d}(v+1) - \int \frac{1}{(v+1)^2}\,\mathrm{d}(v+1)$$

$$= \ln|v+1| + \frac{1}{v+1},$$

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x|, \quad \int \frac{1}{x^2}\,\mathrm{d}x = -\frac{1}{x}$$

one obtains

$$\ln|v+1| + \frac{1}{v+1} = -\ln|x| + C.$$

Converting back to the original variables x and y results in the general solution

$$\ln\left|\frac{y}{x}+1\right| + \ln|x| + \frac{1}{\frac{y}{x}+1} = C, \qquad \text{✍ } \ln a + \ln b = \ln(a\cdot b)$$

$$\therefore \quad \ln|y+x| + \frac{x}{y+x} = C. \qquad \text{✍ General solution}$$

The constant C is determined using the initial condition $y(0)=1$

$$\ln|1+0| + \frac{0}{1+0} = C \quad \Longrightarrow \quad C = 0.$$

The particular solution satisfying $y(0) = 1$ is

$$\ln|y+x| + \frac{x}{y+x} = 0.$$ Particular solution

Example 2.5

Solve $\quad x(\ln x - \ln y)\,\mathrm{d}y - y\,\mathrm{d}x = 0, \quad x > 0, \quad y > 0.$

Dividing both sides of the equation by x gives

$$\left(\ln \frac{x}{y}\right)\mathrm{d}y - \frac{y}{x}\,\mathrm{d}x = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dfrac{y}{x}}{-\ln \dfrac{y}{x}},$$

which is homogeneous. Putting $v = \dfrac{y}{x}$, $v > 0$,

$$y = xv \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x},$$

the equation becomes

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v}{-\ln v} \quad \Longrightarrow \quad x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{v}{\ln v} - v = -v\,\frac{1 + \ln v}{\ln v}.$$

$$\frac{\ln v}{v(1+\ln v)}\,\mathrm{d}v = -\frac{1}{x}\,\mathrm{d}x.$$

Integrating both sides yields

$$\int \frac{\ln v}{v(1+\ln v)}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x + C.$$

Since $\mathrm{d}(\ln v) = \frac{1}{v}\mathrm{d}v$, one has

$$\int \frac{\ln v}{v(1+\ln v)}\,\mathrm{d}v = \int \frac{(1+\ln v)-1}{1+\ln v}\,\mathrm{d}(\ln v)$$

$$= \int \left(1-\frac{1}{1+\ln v}\right)\mathrm{d}(\ln v) = \ln v - \ln|1+\ln v|.$$

Hence,

$$\ln\left|\frac{v}{1+\ln v}\right| = -\ln x + C.$$

Replacing v by the original variables results in the general solution

$$\ln\left|\frac{y/x}{1+\ln(y/x)}\right| + \ln x = C,$$

$$\ln\left|\frac{y}{1+\ln y - \ln x}\right| = \ln C,$$

✍ Since C is an arbitrary constant, it is rewritten as $\ln C$.

$$\therefore \quad \frac{y}{1+\ln y - \ln x} = C.$$

Example 2.6

Solve $$(y+x)\,\mathrm{d}y+(x-y)\,\mathrm{d}x=0.$$

Since $y=-x$ is not a solution, the equation can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x-y}{y+x}=-\frac{1-\dfrac{y}{x}}{\dfrac{y}{x}+1},$$

✍ Dividing both the numerator and denominator by x, $x\neq 0$

which is a homogeneous equation. Letting $v=\dfrac{y}{x}$,

$$y=xv\quad\Longrightarrow\quad\frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x},$$

the equation becomes

$$v + x\frac{dv}{dx} = -\frac{1-v}{v+1},$$

$$x\frac{dv}{dx} = -\frac{1-v}{v+1} - v = -\frac{1-v+v(v+1)}{v+1} = -\frac{v^2+1}{v+1}.$$

Since $v^2+1 \neq 0$, separating the variables gives

$$\frac{v+1}{v^2+1}dv = -\frac{1}{x}dx.$$

Integrating both sides yields

$$\int \frac{v}{v^2+1}dv + \int \frac{1}{v^2+1}dv = -\int \frac{1}{x}dx + C,$$

$$\frac{1}{2}\int \frac{1}{v^2+1}d(v^2+1) + \tan^{-1}v = -\ln|x| + C, \quad \int \frac{1}{x^2+1}dx = \tan^{-1}x$$

$$\frac{1}{2}\ln\left|v^2+1\right| + \tan^{-1}v = -\ln|x| + C.$$

$\int \frac{1}{x}\,dx = \ln|x|$

Replacing v by the original variables x and y results in the general solution

$$\ln\left|\left(\frac{y}{x}\right)^2+1\right| + 2\ln|x| + 2\tan^{-1}\frac{y}{x} = 2C,$$

$a\ln b = \ln b^a$

$\ln a + \ln b = \ln(a\cdot b)$

$$\therefore \quad \ln(y^2+x^2) + 2\tan^{-1}\frac{y}{x} = C.$$

$2C$ is renamed as C.