8.4 TRIGONOMETRIC INTEGRALS

1. $\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} (\sin^2 x)^2 \sin x\, dx = \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x\, dx = \int_0^{\pi/2} (1 - 2\cos^2 x + \cos^4 x)\sin x\, dx$
$= \int_0^{\pi/2} \sin x\, dx - \int_0^{\pi/2} 2\cos^2 x \sin x\, dx + \int_0^{\pi/2} \cos^4 x \sin x\, dx = \left[-\cos x + 2\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right]_0^{\pi/2}$
$= (0) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right) = \frac{8}{15}$

2. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right) dx$ (using Exercise 1) $= \int_0^{\pi} \sin\left(\frac{x}{2}\right) dx - \int_0^{\pi} 2\cos^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx + \int_0^{\pi} \cos^4\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx$
$= \left[-2\cos\left(\frac{x}{2}\right) + \frac{4}{3}\cos^3\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right)\right]_0^{\pi} = (0) - \left(-2 + \frac{4}{3} - \frac{2}{5}\right) = \frac{16}{15}$

3. $\int_{-\pi/2}^{\pi/2} \cos^3 x\, dx = \int_{-\pi/2}^{\pi/2} (\cos^2 x)\cos x\, dx = \int_{-\pi/2}^{\pi/2} (1 - \sin^2 x)\cos x\, dx = \int_{-\pi/2}^{\pi/2} \cos x\, dx - \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx$
$= \left[\sin x - \frac{\sin^3 x}{3}\right]_{-\pi/2}^{\pi/2} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$

4. $\int_0^{\pi/6} 3\cos^5 3x\, dx = \int_0^{\pi/6} (\cos^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1 - \sin^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1 - 2\sin^2 3x + \sin^4 3x)\cos 3x \cdot 3dx$
$= \int_0^{\pi/6} \cos 3x \cdot 3dx - 2\int_0^{\pi/6} \sin^2 3x \cos 3x \cdot 3dx + \int_0^{\pi/6} \sin^4 3x \cos 3x \cdot 3dx = \left[\sin 3x - 2\frac{\sin^3 3x}{3} + \frac{\sin^5 3x}{5}\right]_0^{\pi/6}$
$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0) = \frac{8}{15}$

5. $\int_0^{\pi/2} \sin^7 y\, dy = \int_0^{\pi/2} \sin^6 y \sin y\, dy = \int_0^{\pi/2} (1 - \cos^2 y)^3 \sin y\, dy = \int_0^{\pi/2} \sin y\, dy - 3\int_0^{\pi/2} \cos^2 y \sin y\, dy$
$+ 3\int_0^{\pi/2} \cos^4 y \sin y\, dy - \int_0^{\pi/2} \cos^6 y \sin y\, dy = \left[-\cos y + 3\frac{\cos^3 y}{3} - 3\frac{\cos^5 y}{5} + \frac{\cos^7 y}{7}\right]_0^{\pi/2} = (0) - \left(-1 + 1 - \frac{3}{5} + \frac{1}{7}\right) = \frac{16}{35}$

6. $\int_0^{\pi/2} 7\cos^7 t\, dt$ (using Exercise 5) $= 7\left[\int_0^{\pi/2} \cos t\, dt - 3\int_0^{\pi/2} \sin^2 t \cos t\, dt + 3\int_0^{\pi/2} \sin^4 t \cos t\, dt - \int_0^{\pi/2} \sin^6 t \cos t\, dt\right]$
$= 7\left[\sin t - 3\frac{\sin^3 t}{3} + 3\frac{\sin^5 t}{5} - \frac{\sin^7 t}{7}\right]_0^{\pi/2} = 7\left(1 - 1 + \frac{3}{5} - \frac{1}{7}\right) - 7(0) = \frac{16}{5}$

7. $\int_0^{\pi} 8\sin^4 x\, dx = 8\int_0^{\pi} \left(\frac{1 - \cos 2x}{2}\right)^2 dx = 2\int_0^{\pi} (1 - 2\cos 2x + \cos^2 2x) dx = 2\int_0^{\pi} dx - 2\int_0^{\pi} \cos 2x \cdot 2dx + 2\int_0^{\pi} \frac{1 + \cos 4x}{2}\, dx$
$= [2x - 2\sin 2x]_0^{\pi} + \int_0^{\pi} dx + \int_0^{\pi} \cos 4x\, dx = 2\pi + \left[x + \frac{1}{2}\sin 4x\right]_0^{\pi} = 2\pi + \pi = 3\pi$

8. $\int_0^1 8\cos^4 2\pi x\, dx = 8\int_0^1 \left(\frac{1 + \cos 4\pi x}{2}\right)^2 dx = 2\int_0^1 (1 + 2\cos 4\pi x + \cos^2 4\pi x) dx = 2\int_0^1 dx + 4\int_0^1 \cos 4\pi x\, dx + 2\int_0^1 \frac{1 + \cos 8\pi x}{2}\, dx$
$= \left[2x + \frac{1}{\pi}\sin 4\pi x\right]_0^1 + \int_0^1 dx + \int_0^1 \cos 8\pi x\, dx = 2 + \left[x + \frac{1}{8\pi}\sin 8\pi x\right]_0^1 = 2 + 1 = 3$

9. $\int_{-\pi/4}^{\pi/4} 16 \sin^2 x \cos^2 x\, dx = 16\int_{-\pi/4}^{\pi/4} \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) dx = 4\int_{-\pi/4}^{\pi/4} (1 - \cos^2 2x) dx = 4\int_{-\pi/4}^{\pi/4} dx - 4\int_{-\pi/4}^{\pi/4} \left(\frac{1 + \cos 4x}{2}\right) dx$
$= [4x]_{-\pi/4}^{\pi/4} - 2\int_{-\pi/4}^{\pi/4} dx - 2\int_{-\pi/4}^{\pi/4} \cos 4x\, dx = \pi + \pi - \left[2x + \frac{\sin 4x}{2}\right]_{-\pi/4}^{\pi/4} = 2\pi - \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi$

10. $\int_0^{\pi} 8 \sin^4 y \cos^2 y\, dy = 8\int_0^{\pi} \left(\frac{1 - \cos 2y}{2}\right)^2 \left(\frac{1 + \cos 2y}{2}\right) dy = \int_0^{\pi} dy - \int_0^{\pi} \cos 2y\, dy - \int_0^{\pi} \cos^2 2y\, dy + \int_0^{\pi} \cos^3 2y\, dy$
$= \left[y - \frac{1}{2}\sin 2y\right]_0^{\pi} - \int_0^{\pi} \left(\frac{1 + \cos 4y}{2}\right) dy + \int_0^{\pi} (1 - \sin^2 2y)\cos 2y\, dy = \pi - \frac{1}{2}\int_0^{\pi} dy - \frac{1}{2}\int_0^{\pi} \cos 4y\, dy + \int_0^{\pi} \cos 2y\, dy$
$- \int_0^{\pi} \sin^2 2y \cos 2y\, dy = \pi + \left[-\frac{1}{2}y - \frac{1}{8}\sin 4y + \frac{1}{2}\sin 2y - \frac{1}{2} \cdot \frac{\sin^3 2y}{3}\right]_0^{\pi} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

11. $\int_0^{\pi/2} 35\sin^4 x\cos^3 x\,dx = \int_0^{\pi/2} 35\sin^4 x(1-\sin^2 x)\cos x\,dx = 35\int_0^{\pi/2}\sin^4 x\cos x\,dx - 35\int_0^{\pi/2}\sin^6 x\cos x\,dx$
$= \left[35\frac{\sin^5 x}{5} - 35\frac{\sin^7 x}{7}\right]_0^{\pi/2} = (7-5)-(0) = 2$

12. $\int_0^{\pi}\cos^2 2x\sin 2x\,dx = \left[-\frac{1}{2}\frac{\cos^3 2x}{3}\right]_0^{\pi} = -\frac{1}{6}+\frac{1}{6} = 0$

13. $\int_0^{\pi/4} 8\cos^3 2\theta\sin 2\theta\,d\theta = \left[8\left(-\frac{1}{2}\right)\frac{\cos^4 2\theta}{4}\right]_0^{\pi/4} = \left[-\cos^4 2\theta\right]_0^{\pi/4} = (0)-(-1) = 1$

14. $\int_0^{\pi/2}\sin^2 2\theta\cos^3 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta(1-\sin^2 2\theta)\cos 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta\cos 2\theta\,d\theta - \int_0^{\pi/2}\sin^4 2\theta\cos 2\theta\,d\theta$
$= \left[\frac{1}{2}\cdot\frac{\sin^3 2\theta}{3} - \frac{1}{2}\cdot\frac{\sin^5 2\theta}{5}\right]_0^{\pi/2} = 0$

15. $\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi}\left|\sin\frac{x}{2}\right|dx = \int_0^{2\pi}\sin\frac{x}{2}\,dx = \left[-2\cos\frac{x}{2}\right]_0^{2\pi} = 2+2 = 4$

16. $\int_0^{\pi}\sqrt{1-\cos 2x}\,dx = \int_0^{\pi}\sqrt{2}\,|\sin 2x\,|dx = \int_0^{\pi}\sqrt{2}\sin 2x\,dx = \left[-\sqrt{2}\cos 2x\right]_0^{\pi} = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

17. $\int_0^{\pi}\sqrt{1-\sin^2 t}\,dt = \int_0^{\pi}|\cos t\,|dt = \int_0^{\pi/2}\cos t\,dt - \int_{\pi/2}^{\pi}\cos t\,dt = [\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{\pi} = 1-0-0+1 = 2$

18. $\int_0^{\pi}\sqrt{1-\cos^2\theta}\,d\theta = \int_0^{\pi}|\sin\theta\,|d\theta = \int_0^{\pi}\sin\theta\,d\theta = [-\cos\theta]_0^{\pi} = 1+1 = 2$

19. $\int_{-\pi/4}^{\pi/4}\sqrt{1+\tan^2 x}\,dx = \int_{-\pi/4}^{\pi/4}|\sec x\,|dx = \int_{-\pi/4}^{\pi/4}\sec x\,dx = [\ln|\sec x+\tan x\,|]_{-\pi/4}^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right)$
$= \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\ln\left(1+\sqrt{2}\right)$

20. $\int_{-\pi/4}^{\pi/4}\sqrt{\sec^2 x-1}\,dx = \int_{-\pi/4}^{\pi/4}|\tan x\,|dx = -\int_{-\pi/4}^{0}\tan x\,dx + \int_0^{\pi/4}\tan x\,dx = [-\ln|\sec x\,|]_{-\pi/4}^{0} + [-\ln|\sec x\,|]_0^{\pi/4}$
$= -\ln(1)+\ln\sqrt{2}+\ln\sqrt{2}-\ln(1) = 2\ln\sqrt{2} = \ln 2$

21. $\int_0^{\pi/2}\theta\sqrt{1-\cos 2\theta}\,d\theta = \int_0^{\pi/2}\theta\sqrt{2}\,|\sin\theta\,|\,d\theta = \sqrt{2}\int_0^{\pi/2}\theta\sin\theta\,d\theta = \sqrt{2}\,[-\theta\cos\theta+\sin\theta]_0^{\pi/2} = \sqrt{2}(1) = \sqrt{2}$

22. $\int_{-\pi}^{\pi}(1-\cos^2 t)^{3/2}\,dt = \int_{-\pi}^{\pi}(\sin^2 t)^{3/2}\,dt = \int_{-\pi}^{\pi}|\sin^3 t|\,dt = -\int_{-\pi}^{0}\sin^3 t\,dt + \int_0^{\pi}\sin^3 t\,dt = -\int_{-\pi}^{0}(1-\cos^2 t)\sin t\,dt$
$+ \int_0^{\pi}(1-\cos^2 t)\sin t\,dt = -\int_{-\pi}^{0}\sin t\,dt + \int_{-\pi}^{0}\cos^2 t\sin t\,dt + \int_0^{\pi}\sin t\,dt - \int_0^{\pi}\cos^2 t\sin t\,dt = \left[\cos t - \frac{\cos^3 t}{3}\right]_{-\pi}^{0}$
$+ \left[-\cos t + \frac{\cos^3 t}{3}\right]_0^{\pi} = \left(1-\frac{1}{3}+1-\frac{1}{3}\right) + \left(1-\frac{1}{3}+1-\frac{1}{3}\right) = \frac{8}{3}$

23. $\int_{-\pi/3}^{0} 2\sec^3 x\,dx$; $u = \sec x$, $du = \sec x\tan x\,dx$, $dv = \sec^2 x\,dx$, $v = \tan x$;
$\int_{-\pi/3}^{0} 2\sec^3 x\,dx = [2\sec x\tan x]_{-\pi/3}^{0} - 2\int_{-\pi/3}^{0}\sec x\tan^2 x\,dx = 2\cdot 1\cdot 0 - 2\cdot 2\cdot\sqrt{3} - 2\int_{-\pi/3}^{0}\sec x\,(\sec^2 x-1)dx$
$= 4\sqrt{3} - 2\int_{-\pi/3}^{0}\sec^3 x\,dx + 2\int_{-\pi/3}^{0}\sec x\,dx$; $2\int_{-\pi/3}^{0} 2\sec^3 x\,dx = 4\sqrt{3} + [2\ln|\sec x+\tan x|]_{-\pi/3}^{0}$
$2\int_{-\pi/3}^{0} 2\sec^3 x\,dx = 4\sqrt{3} + 2\ln|1+0| - 2\ln|2-\sqrt{3}| = 4\sqrt{3} - 2\ln\left(2-\sqrt{3}\right)$
$\int_{-\pi/3}^{0} 2\sec^3 x\,dx = 2\sqrt{3} - \ln\left(2-\sqrt{3}\right)$

24. $\int e^x \sec^3(e^x)dx$; $u = \sec(e^x)$, $du = \sec(e^x)\tan(e^x)e^x dx$, $dv = \sec^2(e^x)e^x dx$, $v = \tan(e^x)$.

$$\int e^x \sec^3(e^x)\,dx = \sec(e^x)\tan(e^x) - \int \sec(e^x)\tan^2(e^x)e^x dx$$
$$= \sec(e^x)\tan(e^x) - \int \sec(e^x)(\sec^2(e^x) - 1)e^x dx$$
$$= \sec(e^x)\tan(e^x) - \int \sec^3(e^x)e^x dx + \int \sec(e^x)e^x dx$$
$$2\int e^x \sec^3(e^x)\,dx = \sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right| + C$$
$$\int e^x \sec^3(e^x)\,dx = \tfrac{1}{2}\left(\sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right|\right) + C$$

25. $\int_0^{\pi/4} \sec^4\theta\, d\theta = \int_0^{\pi/4} (1 + \tan^2\theta)\sec^2\theta\, d\theta = \int_0^{\pi/4} \sec^2\theta\, d\theta + \int_0^{\pi/4} \tan^2\theta \sec^2\theta\, d\theta = \left[\tan\theta + \frac{\tan^3\theta}{3}\right]_0^{\pi/4}$
$= \left(1 + \frac{1}{3}\right) - (0) = \frac{4}{3}$

26. $\int_0^{\pi/12} 3\sec^4(3x)\,dx = \int_0^{\pi/12} (1 + \tan^2(3x))\sec^2(3x)3dx = \int_0^{\pi/} \sec^2(3x)3dx + \int_0^{\pi/12} \tan^2(3x)\sec^2(3x)3dx$
$= \left[\tan(3x) + \frac{\tan^3(3x)}{3}\right]_0^{\pi/12} = \left(1 + \frac{1}{3}\right) - (0) = \frac{4}{3}$

27. $\int_{\pi/4}^{\pi/2} \csc^4\theta\, d\theta = \int_{\pi/4}^{\pi/2} (1 + \cot^2\theta)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \cot^2\theta\csc^2\theta\, d\theta = \left[-\cot\theta - \frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2}$
$= (0) - \left(-1 - \frac{1}{3}\right) = \frac{4}{3}$

28. $\int_{\pi/2}^{\pi} 3\csc^4\frac{\theta}{2}\, d\theta = 3\int_{\pi/2}^{\pi} \left(1 + \cot^2\frac{\theta}{2}\right)\csc^2\frac{\theta}{2}\, d\theta = 3\int_{\pi/2}^{\pi} \csc^2\frac{\theta}{2}\, d\theta + 3\int_{\pi/2}^{\pi} \cot^2\frac{\theta}{2}\csc^2\frac{\theta}{2}\, d\theta = \left[-6\cot\frac{\theta}{2} - 6\frac{\cot^3\frac{\theta}{2}}{3}\right]_{\pi/2}^{\pi}$
$= (-6 \cdot 0 - 2 \cdot 0) - (-6 \cdot 1 - 2 \cdot 1) = 8$

29. $\int_0^{\pi/4} 4\tan^3 x\, dx = 4\int_0^{\pi/4} (\sec^2 x - 1)\tan x\, dx = 4\int_0^{\pi/4} \sec^2 x \tan x\, dx - 4\int_0^{\pi/4} \tan x\, dx = \left[4\frac{\tan^2 x}{2} - 4\ln|\sec x|\right]_0^{\pi/4}$
$= 2(1) - 4\ln\sqrt{2} - 2 \cdot 0 + 4\ln 1 = 2 - 2\ln 2$

30. $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\, dx = 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)\tan^2 x\, dx = 6\int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} \tan^2 x\, dx$
$= 6\int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)dx = \left[6\frac{\tan^3 x}{3}\right]_{-\pi/4}^{\pi/4} - 6\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx + 6\int_{-\pi/4}^{\pi/4} dx$
$= 2(1 - (-1)) - [6\tan x]_{-\pi/4}^{\pi/4} + [6x]_{-\pi/4}^{\pi/4} = 4 - 6(1 - (-1)) + \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi - 8$

31. $\int_{\pi/6}^{\pi/3} \cot^3 x\, dx = \int_{\pi/6}^{\pi/3} (\csc^2 x - 1)\cot x\, dx = \int_{\pi/6}^{\pi/3} \csc^2 x \cot x\, dx - \int_{\pi/6}^{\pi/3} \cot x\, dx = \left[-\frac{\cot^2 x}{2} + \ln|\csc x|\right]_{\pi/6}^{\pi/3}$
$= -\frac{1}{2}\left(\frac{1}{3} - 3\right) + \left(\ln\frac{2}{\sqrt{3}} - \ln 2\right) = \frac{4}{3} - \ln\sqrt{3}$

32. $\int_{\pi/4}^{\pi/2} 8\cot^4 t\, dt = 8\int_{\pi/4}^{\pi/2} (\csc^2 t - 1)\cot^2 t\, dt = 8\int_{\pi/4}^{\pi/2} \csc^2 t \cot^2 t\, dt - 8\int_{\pi/4}^{\pi/2} \cot^2 t\, dt$
$= -8\left[-\frac{\cot^3 t}{3}\right]_{\pi/4}^{\pi/2} - 8\int_{\pi/4}^{\pi/2} (\csc^2 t - 1)dt = -\frac{8}{3}(0 - 1) + [8\cot t]_{\pi/4}^{\pi/2} + [8t]_{\pi/4}^{\pi/2} = \frac{8}{3} + 8(0 - 1) + 4\pi - 2\pi = 2\pi - \frac{16}{3}$

33. $\int_{-\pi}^{0} \sin 3x \cos 2x\, dx = \frac{1}{2}\int_{-\pi}^{0} (\sin x + \sin 5x)\, dx = \frac{1}{2}\left[-\cos x - \frac{1}{5}\cos 5x\right]_{-\pi}^{0} = \frac{1}{2}\left(-1 - \frac{1}{5} - 1 - \frac{1}{5}\right) = -\frac{6}{5}$

34. $\int_0^{\pi/2} \sin 2x \cos 3x\, dx = \frac{1}{2}\int_0^{\pi/2} (\sin(-x) + \sin 5x)\, dx = \frac{1}{2}\left[\cos(-x) - \frac{1}{5}\cos 5x\right]_0^{\pi/2} = \frac{1}{2}(0) - \frac{1}{2}\left(1 - \frac{1}{5}\right) = -\frac{2}{5}$

35. $\int_{-\pi}^{\pi} \sin 3x \sin 3x\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos 6x\, dx = \frac{1}{2}\left[x - \frac{1}{12}\sin 6x\right]_{-\pi}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi$

36. $\int_0^{\pi/2} \sin x \cos x\, dx = \frac{1}{2}\int_0^{\pi/2} (\sin 0 + \sin 2x)\, dx = \frac{1}{2}\int_0^{\pi/2} \sin 2x\, dx = -\frac{1}{4}[\cos 2x]_0^{\pi/2} = -\frac{1}{4}(-1-1) = \frac{1}{2}$

37. $\int_0^{\pi} \cos 3x \cos 4x\, dx = \frac{1}{2}\int_0^{\pi} (\cos(-x) + \cos 7x)\, dx = \frac{1}{2}\left[-\sin(-x) + \frac{1}{7}\sin 7x\right]_0^{\pi} = \frac{1}{2}(0) = 0$

38. $\int_{-\pi/2}^{\pi/2} \cos 7x \cos x\, dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 6x + \cos 8x)\, dx = \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2} = 0$

39. $x = t^{2/3} \Rightarrow t^2 = x^3;\ y = \frac{t^2}{2} \Rightarrow y = \frac{x^3}{2};\ 0 \le t \le 2 \Rightarrow 0 \le x \le 2^{2/3};$
$A = \int_0^{2^{2/3}} 2\pi\left(\frac{x^3}{2}\right)\sqrt{1 + \frac{9}{4}x^4}\, dx;\ \left[\begin{array}{c} u = \frac{9}{4}x^4 \\ du = 9x^3 dx \end{array}\right] \to \frac{\pi}{9}\int_0^{9(2^{2/3})} \sqrt{1+u}\, du = \left[\frac{\pi}{9}\cdot\frac{2}{3}(1+u)^{3/2}\right]_0^{9(2^{2/3})}$
$= \frac{2\pi}{27}\left[\left(1 + 9\left(2^{2/3}\right)\right)^{3/2} - 1\right]$

40. $y = \ln(\cos x);\ y' = \frac{-\sin x}{\cos x} = -\tan x;\ (y')^2 = \tan^2 x;\ \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/3} |\sec x|\, dx = [\ln|\sec x + \tan x|]_0^{\pi/3}$
$= \ln\left(2 + \sqrt{3}\right) - \ln(1+0) = \ln\left(2 + \sqrt{3}\right)$

41. $y = \ln(\sec x);\ y' = \frac{\sec x \tan x}{\sec x} = \tan x; (y')^2 = \tan^2 x;\ \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} |\sec x|\, dx = [\ln|\sec x + \tan x|]_0^{\pi/4}$
$= \ln\left(\sqrt{2} + 1\right) - \ln(0+1) = \ln\left(\sqrt{2} + 1\right)$

42. $M = \int_{-\pi/4}^{\pi/4} \sec x\, dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln|\sqrt{2} - 1| = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$
$\overline{y} = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}\int_{-\pi/4}^{\pi/4} \frac{\sec^2 x}{2}\, dx = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}[\tan x]_{-\pi/4}^{\pi/4} = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}(1 - (-1)) = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}$
$\Rightarrow (\overline{x}, \overline{y}) = \left(0, \left(\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{-1}\right)$

43. $V = \pi\int_0^{\pi} \sin^2 x\, dx = \pi\int_0^{\pi} \frac{1-\cos 2x}{2}\, dx = \frac{\pi}{2}\int_0^{\pi} dx - \frac{\pi}{2}\int_0^{\pi} \cos 2x\, dx = \frac{\pi}{2}[x]_0^{\pi} - \frac{\pi}{4}[\sin 2x]_0^{\pi} = \frac{\pi}{2}(\pi - 0) - \frac{\pi}{4}(0-0) = \frac{\pi^2}{2}$

44. $A = \int_0^{\pi} \sqrt{1 + \cos 4x}\, dx = \int_0^{\pi} \sqrt{2}\,|\cos 2x| dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx - \sqrt{2}\int_{\pi/4}^{3\pi/4} \cos 2x\, dx + \sqrt{2}\int_{3\pi/4}^{\pi} \cos 2x\, dx$
$= \frac{\sqrt{2}}{2}[\sin 2x]_0^{\pi/4} - \frac{\sqrt{2}}{2}[\sin 2x]_{\pi/4}^{3\pi/4} + \frac{\sqrt{2}}{2}[\sin 2x]_{3\pi/4}^{\pi} = \frac{\sqrt{2}}{2}(1-0) - \frac{\sqrt{2}}{2}(-1-1) + \frac{\sqrt{2}}{2}(0+1) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

45. (a) $m^2 \ne n^2 \Rightarrow m + n \ne 0$ and $m - n \ne 0 \Rightarrow \int_k^{k+2\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_k^{k+2\pi} [\cos(m-n)x - \cos(m+n)x] dx$
$= \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x - \frac{1}{m+n}\sin(m+n)x\right]_k^{k+2\pi}$
$= \frac{1}{2}\left(\frac{1}{m-n}\sin((m-n)(k+2\pi)) - \frac{1}{m+n}\sin((m+n)(k+2\pi))\right) - \frac{1}{2}\left(\frac{1}{m-n}\sin((m-n)k) - \frac{1}{m+n}\sin((m+n)k)\right)$
$= \frac{1}{2(m-n)}\sin((m-n)k) - \frac{1}{2(m+n)}\sin((m+n)k) - \frac{1}{2(m-n)}\sin((m-n)k) + \frac{1}{2(m+n)}\sin((m+n)k) = 0$
$\Rightarrow$ sin mx and sin nx are orthogonal.

(b) Same as part since $\frac{1}{2}\int_k^{k+2\pi} \cos 0\, dx = \pi$. $m^2 \ne n^2 \Rightarrow m + n \ne 0$ and $m - n \ne 0 \Rightarrow \int_k^{k+2\pi} \cos mx \cos nx\, dx$
$= \frac{1}{2}\int_k^{k+2\pi} [\cos(m-n)x + \cos(m+n)x] dx = \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x + \frac{1}{m+n}\sin(m+n)x\right]_k^{k+2\pi}$
$= \frac{1}{2(m-n)}\sin((m-n)(k+2\pi)) + \frac{1}{2(m+n)}\sin((m+n)(k+2\pi)) - \frac{1}{2(m-n)}\sin((m-n)k) - \frac{1}{2(m+n)}\sin((m+n)k)$
$= \frac{1}{2(m-n)}\sin((m-n)k) + \frac{1}{2(m+n)}\sin((m+n)k) - \frac{1}{2(m-n)}\sin((m-n)k) - \frac{1}{2(m+n)}\sin((m+n)k) = 0$