40. $I = \int x^n \sin x \, dx$; $[u = x^n, \, du = nx^{n-1} \, dx; \, dv = \sin x \, dx, \, v = -\cos x]$

   $\Rightarrow I = -x^n \cos x + \int n x^{n-1} \cos x \, dx$

41. $I = \int x^n e^{ax} \, dx$; $\left[u = x^n, \, du = nx^{n-1} \, dx; \, dv = e^{ax} \, dx, \, v = \frac{1}{a}e^{ax}\right]$

   $\Rightarrow I = \frac{x^n e^{ax}}{a} e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \, a \neq 0$

42. $I = \int (\ln x)^n \, dx$; $\left[u = (\ln x)^n, \, du = \frac{n(\ln x)^{n-1}}{x} \, dx; \, dv = 1 \, dx, \, v = x\right]$

   $\Rightarrow I = x(\ln x)^n - \int n(\ln x)^{n-1} \, dx$

43. $\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \sin y \, dy = x \sin^{-1} x + \cos y + C = x \sin^{-1} x + \cos(\sin^{-1} x) + C$

44. $\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \tan y \, dy = x \tan^{-1} x + \ln|\cos y| + C = x \tan^{-1} x + \ln|\cos(\tan^{-1} x)| + C$

45. $\int \sec^{-1} x \, dx = x \sec^{-1} x - \int \sec y \, dy = x \sec^{-1} x - \ln|\sec y + \tan y| + C$

   $= x \sec^{-1} x - \ln|\sec(\sec^{-1} x) + \tan(\sec^{-1} x)| + C = x \sec^{-1} x - \ln\left|x + \sqrt{x^2 - 1}\right| + C$

46. $\int \log_2 x \, dx = x \log_2 x - \int 2^y \, dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{x}{\ln 2} + C$

47. Yes, $\cos^{-1} x$ is the angle whose cosine is x which implies $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$.

48. Yes, $\tan^{-1} x$ is the angle whose tangent is x which implies $\sec(\tan^{-1} x) = \sqrt{1 + x^2}$.

49. (a) $\int \sinh^{-1} x \, dx = x \sinh^{-1} x - \int \sinh y \, dy = x \sinh^{-1} x - \cosh y + C = x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C$;

   check: $d\left[x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C\right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \sinh(\sinh^{-1} x) \frac{1}{\sqrt{1+x^2}}\right] dx$

   $= \sinh^{-1} x \, dx$

   (b) $\int \sinh^{-1} x \, dx = x \sinh^{-1} x - \int x \left(\frac{1}{\sqrt{1+x^2}}\right) dx = x \sinh^{-1} x - \frac{1}{2} \int (1 + x^2)^{-1/2} 2x \, dx$

   $= x \sinh^{-1} x - (1 + x^2)^{1/2} + C$

   check: $d\left[x \sinh^{-1} x - (1 + x^2)^{1/2} + C\right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}\right] dx = \sinh^{-1} x \, dx$

50. (a) $\int \tanh^{-1} x \, dx = x \tanh^{-1} x - \int \tanh y \, dy = x \tanh^{-1} x - \ln|\cosh y| + C$

   $= x \tanh^{-1} x - \ln|\cosh(\tanh^{-1} x)| + C$;

   check: $d\left[x \tanh^{-1} x - \ln|\cosh(\tanh^{-1} x)| + C\right] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{\sinh(\tanh^{-1} x)}{\cosh(\tanh^{-1} x)} \frac{1}{1-x^2}\right] dx$

   $= \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right] dx = \tanh^{-1} x \, dx$

   (b) $\int \tanh^{-1} x \, dx = x \tanh^{-1} x - \int \frac{x}{1-x^2} \, dx = x \tanh^{-1} x - \frac{1}{2} \int \frac{2x}{1-x^2} \, dx = x \tanh^{-1} x + \frac{1}{2} \ln|1 - x^2| + C$

   check: $d\left[x \tanh^{-1} x + \frac{1}{2} \ln|1 - x^2| + C\right] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right] dx = \tanh^{-1} x \, dx$

8.3 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

1. $\frac{5x - 13}{(x - 3)(x - 2)} = \frac{A}{x - 3} + \frac{B}{x - 2} \Rightarrow 5x - 13 = A(x - 2) + B(x - 3) = (A + B)x - (2A + 3B)$

   $\Rightarrow \left.\begin{array}{c} A + B = 5 \\ 2A + 3B = 13 \end{array}\right\} \Rightarrow -B = (10 - 13) \Rightarrow B = 3 \Rightarrow A = 2$; thus, $\frac{5x - 13}{(x - 3)(x - 2)} = \frac{2}{x - 3} + \frac{3}{x - 2}$

2. $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x-7 = A(x-1) + B(x-2) = (A+B)x - (A+2B)$

$\Rightarrow \left. \begin{array}{l} A+B=5 \\ A+2B=7 \end{array} \right\} \Rightarrow B=2 \Rightarrow A=3;$ thus, $\frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$

3. $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1)+B = Ax + (A+B) \Rightarrow \left. \begin{array}{l} A=1 \\ A+B=4 \end{array} \right\} \Rightarrow A=1$ and $B=3;$

thus, $\frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1)+B = Ax + (-A+B) \Rightarrow \left. \begin{array}{l} A=2 \\ -A+B=2 \end{array} \right\}$

$\Rightarrow A=2$ and $B=4;$ thus, $\frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$

5. $\frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1) + B(z-1) + Cz^2 \Rightarrow z+1 = (A+C)z^2 + (-A+B)z - B$

$\Rightarrow \left. \begin{array}{l} A+C=0 \\ -A+B=1 \\ -B=1 \end{array} \right\} \Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2;$ thus, $\frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$

6. $\frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2) + B(z-3) = (A+B)z + (2A-3B)$

$\Rightarrow \left. \begin{array}{l} A+B=0 \\ 2A-3B=1 \end{array} \right\} \Rightarrow -5B=1 \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5};$ thus, $\frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$

7. $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$

$\Rightarrow 5t+2 = A(t-2) + B(t-3) = (A+B)t + (-2A-3B) \Rightarrow \left. \begin{array}{l} A+B=5 \\ -2A-3B=2 \end{array} \right\} \Rightarrow -B = (10+2) = 12$

$\Rightarrow B=-12 \Rightarrow A=17;$ thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$

8. $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)}$ (after long division); $\frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$

$\Rightarrow -9t^2+9 = At(t^2+9) + B(t^2+9) + (Ct+D)t^2 = (A+C)t^3 + (B+D)t^2 + 9At + 9B$

$\Rightarrow \left. \begin{array}{l} A+C=0 \\ B+D=-9 \\ 9A=0 \\ 9B=9 \end{array} \right\} \Rightarrow A=0 \Rightarrow C=0; B=1 \Rightarrow D=-10;$ thus, $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$

9. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x) + B(1-x); x=1 \Rightarrow A=\frac{1}{2}; x=-1 \Rightarrow B=\frac{1}{2};$

$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}[\ln|1+x| - \ln|1-x|] + C$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + Bx; x=0 \Rightarrow A=\frac{1}{2}; x=-2 \Rightarrow B=-\frac{1}{2};$

$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}[\ln|x| - \ln|x+2|] + C$

11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1) + B(x+6); x=1 \Rightarrow B=\frac{5}{7}; x=-6 \Rightarrow A=\frac{-2}{-7}=\frac{2}{7};$

$\int \frac{x+4}{x^2+5x-6} dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln|(x+6)^2(x-1)^5| + C$

12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3) + B(x-4); x=3 \Rightarrow B=\frac{7}{-1}=-7; x=4 \Rightarrow A=\frac{9}{1}=9;$

$\int \frac{2x+1}{x^2-7x+12} dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

13. $\frac{y}{y^2 - 2y - 3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3); y = -1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4}; y = 3 \Rightarrow A = \frac{3}{4};$

$\int_4^8 \frac{y \, dy}{y^2 - 2y - 3} = \frac{3}{4} \int_4^8 \frac{dy}{y-3} + \frac{1}{4} \int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4} \ln|y-3| + \frac{1}{4} \ln|y+1| \right]_4^8 = \left(\frac{3}{4} \ln 5 + \frac{1}{4} \ln 9 \right) - \left(\frac{3}{4} \ln 1 + \frac{1}{4} \ln 5 \right)$

$= \frac{1}{2} \ln 5 + \frac{1}{2} \ln 3 = \frac{\ln 15}{2}$

14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y + 4 = A(y+1) + By; y = 0 \Rightarrow A = 4; y = -1 \Rightarrow B = \frac{3}{-1} = -3;$

$\int_{1/2}^1 \frac{y+4}{y^2+y} \, dy = 4 \int_{1/2}^1 \frac{dy}{y} - 3 \int_{1/2}^1 \frac{dy}{y+1} = [4 \ln|y| - 3 \ln|y+1|]_{1/2}^1 = (4 \ln 1 - 3 \ln 2) - \left(4 \ln \frac{1}{2} - 3 \ln \frac{3}{2} \right)$

$= \ln \frac{1}{8} - \ln \frac{1}{16} + \ln \frac{27}{8} = \ln \left(\frac{27}{8} \cdot \frac{1}{8} \cdot 16 \right) = \ln \frac{27}{4}$

15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2); t = 0 \Rightarrow A = -\frac{1}{2}; t = -2$

$\Rightarrow B = \frac{1}{6}; t = 1 \Rightarrow C = \frac{1}{3}; \int \frac{dt}{t^3+t^2-2t} = -\frac{1}{2} \int \frac{dt}{t} + \frac{1}{6} \int \frac{dt}{t+2} + \frac{1}{3} \int \frac{dt}{t-1}$

$= -\frac{1}{2} \ln|t| + \frac{1}{6} \ln|t+2| + \frac{1}{3} \ln|t-1| + C$

16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) + Cx(x+2); x = 0 \Rightarrow A = \frac{3}{-8}; x = -2$

$\Rightarrow B = \frac{1}{16}; x = 2 \Rightarrow C = \frac{5}{16}; \int \frac{x+3}{2x^3-8x} \, dx = -\frac{3}{8} \int \frac{dx}{x} + \frac{1}{16} \int \frac{dx}{x+2} + \frac{5}{16} \int \frac{dx}{x-2}$

$= -\frac{3}{8} \ln|x| + \frac{1}{16} \ln|x+2| + \frac{5}{16} \ln|x-2| + C = \frac{1}{16} \ln \left| \frac{(x-2)^5(x+2)}{x^6} \right| + C$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1) + B$

$= Ax + (A+B) \Rightarrow A = 3, A+B = 2 \Rightarrow A = 3, B = -1; \int_0^1 \frac{x^3 \, dx}{x^2+2x+1}$

$= \int_0^1 (x-2) \, dx + 3 \int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3 \ln|x+1| + \frac{1}{x+1} \right]_0^1$

$= \left(\frac{1}{2} - 2 + 3 \ln 2 + \frac{1}{2} \right) - (1) = 3 \ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1) + B$

$= Ax + (-A+B) \Rightarrow A = 3, -A+B = -2 \Rightarrow A = 3, B = 1; \int_{-1}^0 \frac{x^3 \, dx}{x^2-2x+1}$

$= \int_{-1}^0 (x+2) \, dx + 3 \int_{-1}^0 \frac{dx}{x-1} + \int_{-1}^0 \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3 \ln|x-1| - \frac{1}{x-1} \right]_{-1}^0$

$= \left(0 + 0 + 3 \ln 1 - \frac{1}{(-1)} \right) - \left(\frac{1}{2} - 2 + 3 \ln 2 - \frac{1}{(-2)} \right) = 2 - 3 \ln 2$

19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$

$x = -1 \Rightarrow C = \frac{1}{4}; x = 1 \Rightarrow D = \frac{1}{4};$ coefficient of $x^3 = A + B \Rightarrow A + B = 0;$ constant $= A - B + C + D$

$\Rightarrow A - B + C + D = 1 \Rightarrow A - B = \frac{1}{2};$ thus, $A = \frac{1}{4} \Rightarrow B = -\frac{1}{4}; \int \frac{dx}{(x^2-1)^2}$

$= \frac{1}{4} \int \frac{dx}{x+1} - \frac{1}{4} \int \frac{dx}{x-1} + \frac{1}{4} \int \frac{dx}{(x+1)^2} + \frac{1}{4} \int \frac{dx}{(x-1)^2} = \frac{1}{4} \ln \left| \frac{x+1}{x-1} \right| - \frac{x}{2(x^2-1)} + C$

20. $\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1); x = -1$

$\Rightarrow C = -\frac{1}{2}; x = 1 \Rightarrow A = \frac{1}{4};$ coefficient of $x^2 = A + B \Rightarrow A + B = 1 \Rightarrow B = \frac{3}{4}; \int \frac{x^2 \, dx}{(x-1)(x^2+2x+1)}$

$= \frac{1}{4} \int \frac{dx}{x-1} + \frac{3}{4} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{dx}{(x+1)^2} = \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+1| + \frac{1}{2(x+1)} + C$

$= \frac{\ln|(x-1)(x+1)^3|}{4} + \frac{1}{2(x+1)} + C$

21. $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1); x = -1 \Rightarrow A = \frac{1}{2};$ coefficient of x^2

$= A + B \Rightarrow A + B = 0 \Rightarrow B = -\frac{1}{2};$ constant $= A + C \Rightarrow A + C = 1 \Rightarrow C = \frac{1}{2}; \int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$$= \frac{1}{2}\int_0^1 \frac{dx}{x+1} + \frac{1}{2}\int_0^1 \frac{(-x+1)}{x^2+1}\, dx = \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x\right]_0^1$$

$$= \left(\frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\tan^{-1}1\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\tan^{-1}0\right) = \frac{1}{4}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{(\pi + 2\ln 2)}{8}$$

22. $\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1) + (Bt+C)t;\ t=0 \Rightarrow A=4;$ coefficient of t^2

$= A + B \Rightarrow A + B = 3 \Rightarrow B = -1;$ coefficient of $t = C \Rightarrow C = 1;\ \int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\, dt$

$= 4\int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1}\, dt = \left[4\ln|t| - \frac{1}{2}\ln(t^2+1) + \tan^{-1}t\right]_1^{\sqrt{3}}$

$= \left(4\ln\sqrt{3} - \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3}\right) - \left(4\ln 1 - \frac{1}{2}\ln 2 + \tan^{-1}1\right) = 2\ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 - \frac{\pi}{4}$

$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12} = \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}$

23. $\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy + D$

$= Ay^3 + By^2 + (A+C)y + (B+D) \Rightarrow A=0, B=1;\ A+C=2 \Rightarrow C=2;\ B+D=1 \Rightarrow D=0;$

$\int \frac{y^2+2y+1}{(y^2+1)^2}\, dy = \int \frac{1}{y^2+1}\, dy + 2\int \frac{y}{(y^2+1)^2}\, dy = \tan^{-1}y - \frac{1}{y^2+1} + C$

24. $\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)(4x^2+1) + Cx + D$

$= 4Ax^3 + 4Bx^2 + (A+C)x + (B+D);\ A=0, B=2;\ A+C=8 \Rightarrow C=8;\ B+D=2 \Rightarrow D=0;$

$\int \frac{8x^2+8x+2}{(4x^2+1)^2}\, dx = 2\int \frac{dx}{4x^2+1} + 8\int \frac{x\, dx}{(4x^2+1)^2} = \tan^{-1}2x - \frac{1}{4x^2+1} + C$

25. $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3} \Rightarrow 2s+2$

$= (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$

$= [As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B] + C(s^4 - 2s^3 + 2s^2 - 2s + 1) + D(s^3 - s^2 + s - 1)$

$\quad + E(s^2+1)$

$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$\Rightarrow \left. \begin{array}{l} A\ \ \ \ +\ C\ \ \ \ \ \ \ \ \ \ \ \ \ = 0 \\ -3A +\ B - 2C + D\ \ \ \ \ \ = 0 \\ 3A - 3B + 2C - D + E = 0 \\ -A + 3B - 2C + D\ \ \ \ \ = 2 \\ -\ B +\ C - D + E = 2 \end{array} \right\}$ summing all equations $\Rightarrow 2E = 4 \Rightarrow E = 2;$

summing eqs (2) and (3) $\Rightarrow -2B + 2 = 0 \Rightarrow B = 1;$ summing eqs (3) and (4) $\Rightarrow 2A + 2 = 2 \Rightarrow A = 0; C = 0$

from eq (1); then $-1 + 0 - D + 2 = 2$ from eq (5) $\Rightarrow D = -1;$

$\int \frac{2s+2}{(s^2+1)(s-1)^3}\, ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1}s + C$

26. $\frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4+81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s$

$= A(s^4 + 18s^2 + 81) + (Bs^4 + Cs^3 + 9Bs^2 + 9Cs) + Ds^2 + Es$

$= (A+B)s^4 + Cs^3 + (18A+9B+D)s^2 + (9C+E)s + 81A \Rightarrow 81A = 81$ or $A = 1;\ A+B = 1 \Rightarrow B = 0;$

$C = 0;\ 9C+E = 0 \Rightarrow E = 0;\ 18A+9B+D = 0 \Rightarrow D = -18;\ \int \frac{s^4+81}{s(s^2+9)^2}\, ds = \int \frac{ds}{s} - 18\int \frac{s\, ds}{(s^2+9)^2}$

$= \ln|s| + \frac{9}{(s^2+9)} + C$

27. $\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3+5\theta^2+8\theta+4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta + D$

$= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A = 2;\ 2A+B = 5 \Rightarrow B = 1;\ 2A+2B+C = 8 \Rightarrow C = 2;$

$2B+D = 4 \Rightarrow D = 2;\ \int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\, d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)}\, d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2}\, d\theta$

$= \int \frac{2\theta+2}{\theta^2+2\theta+2}\, d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$

$$= \frac{-1}{\theta^2 + 2\theta + 2} + \ln\left(\theta^2 + 2\theta + 2\right) - \tan^{-1}\left(\theta + 1\right) + C$$

28. $\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2 + 1\right)^3} = \frac{A\theta + B}{\theta^2 + 1} + \frac{C\theta + D}{\left(\theta^2 + 1\right)^2} + \frac{E\theta + F}{\left(\theta^2 + 1\right)^3} \Rightarrow \theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1$

$= (A\theta + B)\left(\theta^2 + 1\right)^2 + (C\theta + D)\left(\theta^2 + 1\right) + E\theta + F = (A\theta + B)\left(\theta^4 + 2\theta^2 + 1\right) + \left(C\theta^3 + D\theta^2 + C\theta + D\right) + E\theta + F$

$= \left(A\theta^5 + B\theta^4 + 2A\theta^3 + 2B\theta^2 + A\theta + B\right) + \left(C\theta^3 + D\theta^2 + C\theta + D\right) + E\theta + F$

$= A\theta^5 + B\theta^4 + (2A + C)\theta^3 + (2B + D)\theta^2 + (A + C + E)\theta + (B + D + F) \Rightarrow A = 0; B = 1; 2A + C = -4$

$\Rightarrow C = -4; 2B + D = 2 \Rightarrow D = 0; A + C + E = -3 \Rightarrow E = 1; B + D + F = 1 \Rightarrow F = 0;$

$\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2 + 1\right)^3}\, d\theta = \int \frac{d\theta}{\theta^2 + 1} - 4 \int \frac{\theta\, d\theta}{\left(\theta^2 + 1\right)^2} + \int \frac{\theta\, d\theta}{\left(\theta^2 + 1\right)^3} = \tan^{-1}\theta + 2\left(\theta^2 + 1\right)^{-1} - \frac{1}{4}\left(\theta^2 + 1\right)^{-2} + C$

29. $\frac{2x^3 - 2x^2 + 1}{x^2 - x} = 2x + \frac{1}{x^2 - x} = 2x + \frac{1}{x(x - 1)}; \frac{1}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1} \Rightarrow 1 = A(x - 1) + Bx; x = 0 \Rightarrow A = -1;$

$x = 1 \Rightarrow B = 1; \int \frac{2x^3 - 2x^2 + 1}{x^2 - x} = \int 2x\, dx - \int \frac{dx}{x} + \int \frac{dx}{x - 1} = x^2 - \ln|x| + \ln|x - 1| + C = x^2 + \ln\left|\frac{x - 1}{x}\right| + C$

30. $\frac{x^4}{x^2 - 1} = (x^2 + 1) + \frac{1}{x^2 - 1} = (x^2 + 1) + \frac{1}{(x + 1)(x - 1)}; \frac{1}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1} \Rightarrow 1 = A(x - 1) + B(x + 1);$

$x = -1 \Rightarrow A = -\frac{1}{2}; x = 1 \Rightarrow B = \frac{1}{2}; \int \frac{x^4}{x^2 - 1}\, dx = \int (x^2 + 1)\, dx - \frac{1}{2} \int \frac{dx}{x + 1} + \frac{1}{2} \int \frac{dx}{x - 1}$

$= \frac{1}{3} x^3 + x - \frac{1}{2} \ln|x + 1| + \frac{1}{2} \ln|x - 1| + C = \frac{x^3}{3} + x + \frac{1}{2} \ln\left|\frac{x - 1}{x + 1}\right| + C$

31. $\frac{9x^3 - 3x + 1}{x^3 - x^2} = 9 + \frac{9x^2 - 3x + 1}{x^2(x - 1)}$ (after long division); $\frac{9x^2 - 3x + 1}{x^2(x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1}$

$\Rightarrow 9x^2 - 3x + 1 = Ax(x - 1) + B(x - 1) + Cx^2; x = 1 \Rightarrow C = 7; x = 0 \Rightarrow B = -1; A + C = 9 \Rightarrow A = 2;$

$\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\, dx = \int 9\, dx + 2 \int \frac{dx}{x} - \int \frac{dx}{x^2} + 7 \int \frac{dx}{x - 1} = 9x + 2 \ln|x| + \frac{1}{x} + 7 \ln|x - 1| + C$

32. $\frac{16x^3}{4x^2 - 4x + 1} = (4x + 4) + \frac{12x - 4}{4x^2 - 4x + 1}; \frac{12x - 4}{(2x - 1)^2} = \frac{A}{2x - 1} + \frac{B}{(2x - 1)^2} \Rightarrow 12x - 4 = A(2x - 1) + B$

$\Rightarrow A = 6; -A + B = -4 \Rightarrow B = 2; \int \frac{16x^3}{4x^2 - 4x + 1}\, dx = 4 \int (x + 1)\, dx + 6 \int \frac{dx}{2x - 1} + 2 \int \frac{dx}{(2x - 1)^2}$

$= 2(x + 1)^2 + 3 \ln|2x - 1| - \frac{1}{2x - 1} + C_1 = 2x^2 + 4x + 3 \ln|2x - 1| - (2x - 1)^{-1} + C$, where $C = 2 + C_1$

33. $\frac{y^4 + y^2 - 1}{y^3 + y} = y - \frac{1}{y\left(y^2 + 1\right)}; \frac{1}{y\left(y^2 + 1\right)} = \frac{A}{y} + \frac{By + C}{y^2 + 1} \Rightarrow 1 = A\left(y^2 + 1\right) + (By + C)y = (A + B)y^2 + Cy + A$

$\Rightarrow A = 1; A + B = 0 \Rightarrow B = -1; C = 0; \int \frac{y^4 + y^2 - 1}{y^3 + y}\, dy = \int y\, dy - \int \frac{dy}{y} + \int \frac{y\, dy}{y^2 + 1}$

$= \frac{y^2}{2} - \ln|y| + \frac{1}{2} \ln\left(1 + y^2\right) + C$

34. $\frac{2y^4}{y^3 - y^2 + y - 1} = 2y + 2 + \frac{2}{y^3 - y^2 + y - 1}; \frac{2}{y^3 - y^2 + y - 1} = \frac{2}{\left(y^2 + 1\right)(y - 1)} = \frac{A}{y - 1} + \frac{By + C}{y^2 + 1}$

$\Rightarrow 2 = A\left(y^2 + 1\right) + (By + C)(y - 1) = \left(Ay^2 + A\right) + \left(By^2 + Cy - By - C\right) = (A + B)y^2 + (-B + C)y + (A - C)$

$\Rightarrow A + B = 0, -B + C = 0$ or $C = B, A - C = A - B = 2 \Rightarrow A = 1, B = -1, C = -1;$

$\int \frac{2y^4}{y^3 - y^2 + y - 1}\, dy = 2 \int (y + 1)\, dy + \int \frac{dy}{y - 1} - \int \frac{y}{y^2 + 1}\, dy - \int \frac{dy}{y^2 + 1}$

$= (y + 1)^2 + \ln|y - 1| - \frac{1}{2} \ln\left(y^2 + 1\right) - \tan^{-1} y + C_1 = y^2 + 2y + \ln|y - 1| - \frac{1}{2} \ln\left(y^2 + 1\right) - \tan^{-1} y + C,$

where $C = C_1 + 1$

35. $\int \frac{e^t\, dt}{e^{2t} + 3e^t + 2} = \left[e^t = y\right] \int \frac{dy}{y^2 + 3y + 2} = \int \frac{dy}{y + 1} - \int \frac{dy}{y + 2} = \ln\left|\frac{y + 1}{y + 2}\right| + C = \ln\left(\frac{e^t + 1}{e^t + 2}\right) + C$

36. $\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\, dt = \int \frac{e^{3t} + 2e^t - 1}{e^{2t} + 1} e^t dt; \begin{bmatrix} y = e^t \\ dy = e^t\, dt \end{bmatrix} \rightarrow \int \frac{y^3 + 2y - 1}{y^2 + 1}\, dy = \int \left(y + \frac{y - 1}{y^2 + 1}\right)\, dy = \frac{y^2}{2} + \int \frac{y}{y^2 + 1}\, dy - \int \frac{dy}{y^2 + 1}$

$= \frac{y^2}{2} + \frac{1}{2} \ln\left(y^2 + 1\right) - \tan^{-1} y + C = \frac{1}{2} e^{2t} + \frac{1}{2} \ln\left(e^{2t} + 1\right) - \tan^{-1}\left(e^t\right) + C$