Basic Concepts for nth *Order Linear Equations*

So, let's start things off here with some basic concepts for n^{th} order linear differential equations. The most general n^{th} order linear differential equation is,

$$P_n(t)\,y^{(n)} + P_{n-1}(t)\,y^{(n-1)} + \cdots + P_1(t)\,y' + P_0(t)\,y = G(t) \tag{4}$$

where you'll hopefully recall that,

$$y^{(m)} = \frac{d^m y}{dx^m}$$

Many of the theorems and ideas for this material require that $y^{(n)}$ has a coefficient of 1 and so if we divide out by $P_n(t)$ we get,

$$y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots + p_1(t)\,y' + p_0(t)\,y = g(t) \tag{5}$$

As we might suspect an IVP for an n^{th} order differential equation will require the following n initial conditions.

$$y(t_0) = \overline{y}_0, \qquad y'(t_0) = \overline{y}_1, \qquad \cdots, \qquad y^{(n-1)}(t_0) = \overline{y}_{n-1} \tag{6}$$

Theorem 1

Suppose the functions $p_0, p_1, \ldots, p_{n-1}$ and $g(t)$ are all continuous in some open interval I containing t_0 then there is a unique solution to the IVP given by (2) and (3) and the solution will exist for all t in I.

This theorem is a very natural extension of a similar theorem we saw in the 1st order material.

Next we need to move into a discussion of the n^{th} order linear homogeneous differential equation,

$$y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots + p_1(t)\,y' + p_0(t)\,y = 0 \qquad \textbf{(7)}$$

Let's suppose that we know $y_1(t), y_2(t), \ldots, y_n(t)$ are all solutions to (4) then by the an extension of the Principle of Superposition we know that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

will also be a solution to (4). The real question here is whether or not this will form a general solution to (4).

Theorem 2

Suppose the functions $p_0, p_1, \ldots, p_{n-1}$ are all continuous on the open interval I and further suppose that $y_1(t), y_2(t), \ldots y_n(t)$ are all solutions to (4). If $W(y_1, y_2, \ldots y_n)(t) \neq 0$ for every t in I then $y_1(t), y_2(t), \ldots y_n(t)$ form a **Fundamental Set of Solutions** and the general solution to (4) is,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

As we did back with the 2nd order material we'll define this to be the **Wronskian** and denote it by,

$$W(y_1, y_2, \ldots y_n)(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem 3

Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to (2) and that $y_1(t), y_2(t), \ldots y_n(t)$ are a fundamental set of solutions to the homogeneous differential equation (4) then,

$$Y_1(t) - Y_2(t)$$

is a solution to (4) and it can be written as

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Now, just as we did with the 2[nd] order material if we let $Y(t)$ be the general solution to (2) and if we let $Y_P(t)$ be any solution to (2) then using the result of this theorem we see that we must have,

$$Y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y_P(t) = y_c(t) + Y_P(t)$$

where, $y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$ is called the **complimentary solution** and $Y_P(t)$ is called a **particular solution**.

Homogeneous Equations with Constant Coefficients

So, let's start off with the following differential equation,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Now, assume that solutions to this differential equation will be in the form $y(t) = \mathbf{e}^{rt}$ and plug this into the differential equation and with a little simplification we get,

$$\mathbf{e}^{rt}\left(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0\right) = 0$$

and so in order for this to be zero we'll need to require that

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

This is called the **characteristic polynomial/equation** and its roots/solutions will give us the solutions to the differential equation. We know that, including repeated roots, an n^{th} degree polynomial (which we have here) will have n roots. So, we need to go through all the possibilities that we've got for roots here.

This is where we start to see differences in how we deal with n^{th} order differential equations versus 2^{nd} order differential equations. There are still the three main cases : real distinct roots, repeated roots and complex roots (although these can now also be repeated as well see). In 2^{nd} order differential equations each differential equation could only involve one of these cases. Now, however, that will not necessarily be the case. We could very easily have differential equations that contains each of these cases.

For instance suppose that we have an 9^{th} order differential equation. The complete list of roots could have 3 roots which only occur once in the list (*i.e.* real distinct roots), a root with multiplicity 4 (*i.e.* occurs 4 times in the list) and a set of complex conjugate roots (recall that because the coefficients are all real complex roots will always occur in conjugate pairs).

for repeated roots we just add in a t for each of the solutions

Example 1 Solve the following IVP.

$$y^{(3)} - 5y'' - 22y' + 56y = 0 \qquad y(0) = 1 \quad y'(0) = -2 \quad y''(0) = -4$$

Solution

The characteristic equation is,

$$r^3 - 5r^2 - 22r + 56 = (r+4)(r-2)(r-7) = 0 \qquad \Rightarrow \qquad r_1 = -4,\ r_2 = 2,\ r_3 = 7$$

So we have three real distinct roots here and so the general solution is,

$$y(t) = c_1 \mathbf{e}^{-4t} + c_2 \mathbf{e}^{2t} + c_3 \mathbf{e}^{7t}$$

Differentiating a couple of times and applying the initial conditions gives the following system of equations that we'll need to solve in order to find the coefficients.

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 \\ -2 &= y'(0) = -4c_1 + 2c_2 + 7c_3 \\ -4 &= y''(0) = 16c_1 + 4c_2 + 49c_3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c_1 &= \tfrac{14}{33} \\ c_2 &= \tfrac{13}{15} \\ c_3 &= -\tfrac{16}{55} \end{aligned}$$

The actual solution is then,

$$y(t) = \tfrac{14}{33}\mathbf{e}^{-4t} + \tfrac{13}{15}\mathbf{e}^{2t} - \tfrac{16}{55}\mathbf{e}^{7t}$$

Example 2 Solve the following differential equation.

$$2y^{(4)} + 11y^{(3)} + 18y'' + 4y' - 8y = 0$$

Solution

The characteristic equation is,

$$2r^4 + 11r^3 + 18r^2 + 4r - 8 = (2r - 1)(r + 2)^3 = 0$$

So, we have two roots here, $r_1 = \frac{1}{2}$ and $r_2 = -2$ which is multiplicity of 3. Remember that we'll get three solutions for the second root and after the first we add t's only the solution until we reach three solutions.

The general solution is then,

$$y(t) = c_1 \mathbf{e}^{\frac{1}{2}t} + c_2 \mathbf{e}^{2t} + c_3 t \mathbf{e}^{2t} + c_4 t^2 \mathbf{e}^{2t}$$

Example 3 Solve the following differential equation.

$$y^{(5)} + 12y^{(4)} + 104y^{(3)} + 408y'' + 1156y' = 0$$

Solution

The characteristic equation is,

$$r^5 + 12r^4 + 104r^3 + 408r^2 + 1156r = r\left(r^2 + 6r + 34\right)^2 = 0$$

So, we have one real root $r = 0$ and a pair of complex roots $r = -3 \pm 5i$ each with multiplicity 2. So, the solution for the real root is easy and for the complex roots we'll get a total of 4 solutions, 2 will be the *normal* solutions and two will be the normal solution each multiplied by t.

The general solution is,

$$y(t) = c_1 + c_2\mathbf{e}^{-3t}\cos(5t) + c_3\mathbf{e}^{-3t}\sin(5t) + c_4\,t\,\mathbf{e}^{-3t}\cos(5t) + c_5\,t\,\mathbf{e}^{-3t}\sin(5t)$$

Example 4 Solve the following differential equation.

$$y^{(5)} - 15y^{(4)} + 84y^{(3)} - 220y'' + 275y' - 125y = 0$$

Solution

The characteristic equation is

$$r^5 - 15r^4 + 84r^3 - 220r^2 + 275r - 125 = (r-1)(r-5)^2(r^2 - 4r + 5) = 0$$

In this case we've got one real distinct root, $r = 1$, and double root, $r = 5$, and a pair of complex roots, $r = 2 \pm i$ that only occur once.

The general solution is then,

$$y(t) = c_1\mathbf{e}^{t} + c_2\mathbf{e}^{5t} + c_3\, t\,\mathbf{e}^{5t} + c_4\mathbf{e}^{2t}\cos(t) + c_5\mathbf{e}^{2t}\sin(t)$$

Example 5 Solve the following differential equation.

$$y^{(4)} + 16y = 0$$

Solution

The characteristic equation is

$$r^4 + 16 = 0$$

So, a really simple characteristic equation. However, in order to find the roots we need to

The 4 (and yes there are 4!) 4^{th} roots of -16 can be found by evaluating the following,

$$\sqrt[4]{-16} = (-16)^{\frac{1}{4}} = \sqrt[4]{16}\mathbf{e}^{\left(\frac{\pi}{4}+\frac{\pi k}{2}\right)i} = 2\left(\cos\left(\tfrac{\pi}{4}+\tfrac{\pi k}{2}\right)+i\cos\left(\tfrac{\pi}{4}+\tfrac{\pi k}{2}\right)\right) \qquad k = 0,1,2,3$$

Here are the 4th roots of -16.

$$k=0:\quad 2\left(\cos\left(\tfrac{\pi}{4}\right)+i\sin\left(\tfrac{\pi}{4}\right)\right)=2\left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)=\sqrt{2}+i\sqrt{2}$$

$$k=1:\quad 2\left(\cos\left(\tfrac{3\pi}{4}\right)+i\sin\left(\tfrac{3\pi}{4}+\tfrac{\pi k}{2}\right)\right)=2\left(-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)=-\sqrt{2}+i\sqrt{2}$$

$$k=2:\quad 2\left(\cos\left(\tfrac{5\pi}{4}\right)+i\sin\left(\tfrac{5\pi}{4}\right)\right)=2\left(-\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right)=-\sqrt{2}-i\sqrt{2}$$

$$k=3:\quad 2\left(\cos\left(\tfrac{7\pi}{4}\right)+i\sin\left(\tfrac{7\pi}{4}\right)\right)=2\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right)=\sqrt{2}-i\sqrt{2}$$

So, we have two sets of complex roots : $r=\sqrt{2}\pm i\sqrt{2}$ and $r=-\sqrt{2}\pm i\sqrt{2}$. The general solution is,

$$y(t)=c_1\mathbf{e}^{\sqrt{2}t}\cos\left(\sqrt{2}\,t\right)+c_2\mathbf{e}^{\sqrt{2}t}\sin\left(\sqrt{2}\,t\right)+c_3\mathbf{e}^{-\sqrt{2}t}\cos\left(\sqrt{2}\,t\right)+c_4\mathbf{e}^{-\sqrt{2}t}\sin\left(\sqrt{2}\,t\right)$$

EXAMPLE 2

Find the general solution of

$$y^{(4)} - y = 0. \tag{14}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 7/2, \qquad y'(0) = -4, \qquad y''(0) = 5/2, \qquad y'''(0) = -2 \tag{15}$$

and draw its graph.

Substituting e^{rt} for y, we find that the characteristic equation is

$$r^4 - 1 = (r^2 - 1)(r^2 + 1) = 0.$$

Therefore, the roots are $r = 1, -1, i, -i$, and the general solution of Eq. (14) is

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

If we impose the initial conditions (15), we obtain

$$c_1 = 0, \qquad c_2 = 3, \qquad c_3 = 1/2, \qquad c_4 = -1;$$

thus the solution of the given initial value problem is

$$y = 3e^{-t} + \tfrac{1}{2}\cos t - \sin t. \tag{16}$$

Example 1 Solve the following differential equation.

$$y^{(3)} - 12y'' + 48y' - 64y = 12 - 32\mathbf{e}^{-8t} + 2\mathbf{e}^{4t}$$

Solution

We first need the complimentary solution so the characteristic equation is,

$$r^3 - 12r^2 + 48r - 64 = (r-4)^3 = 0 \qquad \Rightarrow \qquad r = 4 \text{ (multiplicity 3)}$$

We've got a single root of multiplicity 3 so the complimentary solution is,

$$y_c(t) = c_1\mathbf{e}^{4t} + c_2 t\mathbf{e}^{4t} + c_3 t^2\mathbf{e}^{4t}$$

Now, our first guess for a particular solution is,

$$Y_P = A + B\mathbf{e}^{-8t} + C\mathbf{e}^{4t}$$

Notice that the last term in our guess is in the complimentary solution so we'll need to add one at least one t to the third term in our guess. Also notice that multiplying the third term by either t or

t^2 will result in a new term that is still in the complimentary solution and so we'll need to multiply the third term by t^3 in order to get a term that is not contained in the complimentary solution.

Our final guess is then,

$$Y_P = A + B\mathbf{e}^{-8t} + Ct^3\mathbf{e}^{4t}$$

Now all we need to do is take three derivatives of this, plug this into the differential equation and simplify to get (we'll leave it to you to verify the work here),

$$-64A - 1728B\mathbf{e}^{-8t} + 6C\mathbf{e}^{4t} = 12 - 32\mathbf{e}^{-8t} + 2\mathbf{e}^{4t}$$

Setting coefficients equal and solving gives,

$$\begin{aligned} t^0 &: & -64A &= 12 & & & A &= -\tfrac{3}{16} \\ \mathbf{e}^{-8t} &: & -1728B &= -32 & &\Rightarrow & B &= \tfrac{1}{54} \\ \mathbf{e}^{4t} &: & 6C &= 2 & & & C &= \tfrac{1}{3} \end{aligned}$$

A particular solution is then,

$$Y_P = -\tfrac{3}{16} + \tfrac{1}{54}\mathbf{e}^{-8t} + \tfrac{1}{3}t^3\mathbf{e}^{4t}$$

The general solution to this differential equation is then,

$$y(t) = c_1\mathbf{e}^{4t} + c_2 t\mathbf{e}^{4t} + c_3 t^2\mathbf{e}^{4t} - \tfrac{3}{16} + \tfrac{1}{54}\mathbf{e}^{-8t} + \tfrac{1}{3}t^3\mathbf{e}^{4t}$$

EXAMPLE 3

Find a particular solution of

$$y''' - 4y' = t + 3\cos t + e^{-2t}. \tag{8}$$

First we solve the homogeneous equation. The characteristic equation is $r^3 - 4r = 0$, and the roots are $r = 0, \pm 2$; hence

$$y_c(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}.$$

We can write a particular solution of Eq. (8) as the sum of particular solutions of the differential equations

$$y''' - 4y' = t, \qquad y''' - 4y' = 3\cos t, \qquad y''' - 4y' = e^{-2t}.$$

Our initial choice for a particular solution $Y_1(t)$ of the first equation is $A_0 t + A_1$, but a constant is a solution of the homogeneous equation, so we multiply by t. Thus

$$Y_1(t) = t(A_0 t + A_1).$$

For the second equation we choose

$$Y_2(t) = B\cos t + C\sin t,$$

and there is no need to modify this initial choice since $\sin t$ and $\cos t$ are not solutions of the homogeneous equation. Finally, for the third equation, since e^{-2t} is a solution of the homogeneous equation, we assume that

$$Y_3(t) = Ete^{-2t}.$$

The constants are determined by substituting into the individual differential equations; they are $A_0 = -\frac{1}{8}$, $A_1 = 0$, $B = 0$, $C = -\frac{3}{5}$, and $E = \frac{1}{8}$. Hence a particular solution of Eq. (8) is

$$Y(t) = -\tfrac{1}{8}t^2 - \tfrac{3}{5}\sin t + \tfrac{1}{8}te^{-2t}. \tag{9}$$