Writing $A\mathbf{x} = \mathbf{b}$ as a linear combination of the columns of A as in (6), we have

$$x_1 \begin{bmatrix} 3 \\ 4 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -5 \\ 7 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 6 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

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The expression for the linear system $A\mathbf{x} = \mathbf{b}$ as shown in (6), provides an important way to think about solutions of linear systems.

> $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as a linear combination of the columns of the matrix A.

We encounter this approach in Chapter 2.

Key Terms

Dot product (inner product)
Matrix–vector product
Coefficient matrix
Augmented matrix

1.3 Exercises

In Exercises 1 and 2, compute $\mathbf{a} \cdot \mathbf{b}$.

1. (a) $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

(d) $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2. (a) $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

(d) $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

3. Let $\mathbf{a} = \mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ x \end{bmatrix}$. If $\mathbf{a} \cdot \mathbf{b} = 17$, find x.

4. Determine the value of x so that $\mathbf{v} \cdot \mathbf{w} = 0$, where

$$\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ x \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} x \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

5. Determine values of x and y so that $\mathbf{v} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{u} = 0$, where $\mathbf{v} = \begin{bmatrix} x \\ 1 \\ y \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$.

6. Determine values of x and y so that $\mathbf{v} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{u} = 0$, where $\mathbf{v} = \begin{bmatrix} x \\ 1 \\ y \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} x \\ -2 \\ 0 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 0 \\ -9 \\ y \end{bmatrix}$.

7. Let $\mathbf{w} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$. Compute $\mathbf{w} \cdot \mathbf{w}$.

8. Find all values of x so that $\mathbf{u} \cdot \mathbf{u} = 50$, where $\mathbf{u} = \begin{bmatrix} x \\ 3 \\ 4 \end{bmatrix}$.

9. Find all values of x so that $\mathbf{v} \cdot \mathbf{v} = 1$, where $\mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ x \end{bmatrix}$.

10. Let $A = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$.

If $AB = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, find x and y.

Consider the following matrices for Exercises 11 through 15:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad \textit{and} \quad F = \begin{bmatrix} -1 & 2 \\ 0 & 4 \\ 3 & 5 \end{bmatrix}.$$

11. If possible, compute the following:

(a) AB **(b)** BA **(c)** $F^T E$

(d) $CB + D$ **(e)** $AB + D^2$, where $D^2 = DD$

12. If possible, compute the following:

(a) $DA + B$ **(b)** EC **(c)** CE

(d) $EB + F$ **(e)** $FC + D$

13. If possible, compute the following:

(a) $FD - 3B$ **(b)** $AB - 2D$

(c) $F^T B + D$ **(d)** $2F - 3(AE)$

(e) $BD + AE$

14. If possible, compute the following:

(a) $A(BD)$ **(b)** $(AB)D$

(c) $A(C + E)$ **(d)** $AC + AE$

(e) $(2AB)^T$ and $2(AB)^T$ **(f)** $A(C - 3E)$

15. If possible, compute the following:

(a) A^T **(b)** $(A^T)^T$

(c) $(AB)^T$ **(d)** $B^T A^T$

(e) $(C + E)^T B$ and $C^T B + E^T B$

(f) $A(2B)$ and $2(AB)$

16. Let $A = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 4 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 0 & 1 \end{bmatrix}$. If possible, compute the following:

(a) AB^T **(b)** CA^T **(c)** $(BA^T)C$

(d) $A^T B$ **(e)** CC^T **(f)** $C^T C$

(g) $B^T CAA^T$

17. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.

Compute the following entries of AB:

(a) the (1, 2) entry **(b)** the (2, 3) entry

(c) the (3, 1) entry **(d)** the (3, 3) entry

18. If $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, compute DI_2 and $I_2 D$.

19. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}.$$

Show that $AB \neq BA$.

20. If A is the matrix in Example 4 and O is the 3×2 matrix every one of whose entries is zero, compute AO.

In Exercises 21 and 22, let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 4 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 3 & -3 & 4 \\ 4 & 2 & 5 & 1 \end{bmatrix}.$$

21. Using the method in Example 11, compute the following columns of AB:

(a) the first column **(b)** the third column

22. Using the method in Example 11, compute the following columns of AB:

(a) the second column **(b)** the fourth column

23. Let

$$A = \begin{bmatrix} 2 & -3 & 4 \\ -1 & 2 & 3 \\ 5 & -1 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Express $A\mathbf{c}$ as a linear combination of the columns of A.

24. Let

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 4 & 3 \\ 3 & 0 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 4 \end{bmatrix}.$$

Express the columns of AB as linear combinations of the columns of A.

25. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.

(**a**) Verify that $AB = 3\mathbf{a}_1 + 5\mathbf{a}_2 + 2\mathbf{a}_3$, where $\mathbf{a}_j$ is the jth column of A for $j = 1, 2, 3$.

(**b**) Verify that $AB = \begin{bmatrix} (\text{row}_1(A))B \\ (\text{row}_2(A))B \end{bmatrix}$.

26. (**a**) Find a value of r so that $AB^T = 0$, where $A = \begin{bmatrix} r & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -1 \end{bmatrix}$.

(**b**) Give an alternative way to write this product.

27. Find a value of r and a value of s so that $AB^T = 0$, where $A = \begin{bmatrix} 1 & r & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 2 & s \end{bmatrix}$.

28. (**a**) Let A be an $m \times n$ matrix with a row consisting entirely of zeros. Show that if B is an $n \times p$ matrix, then AB has a row of zeros.

(**b**) Let A be an $m \times n$ matrix with a column consisting entirely of zeros and let B be $p \times m$. Show that BA has a column of zeros.

29. Let $A = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 0 \end{bmatrix}$ with $\mathbf{a}_j =$ the jth column of A, $j = 1, 2, 3$. Verify that

$$A^T A = \begin{bmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \mathbf{a}_1^T\mathbf{a}_3 \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \mathbf{a}_2^T\mathbf{a}_3 \\ \mathbf{a}_3^T\mathbf{a}_1 & \mathbf{a}_3^T\mathbf{a}_2 & \mathbf{a}_3^T\mathbf{a}_3 \end{bmatrix}.$$

30. Consider the following linear system:

$$\begin{aligned} 2x_1 + 3x_2 - 3x_3 + x_4 + x_5 &= 7 \\ 3x_1 \quad\quad + 2x_3 \quad\quad + 3x_5 &= -2 \\ 2x_1 + 3x_2 \quad\quad - 4x_4 \quad\quad &= 3 \\ x_3 + x_4 + x_5 &= 5. \end{aligned}$$

(**a**) Find the coefficient matrix.

(**b**) Write the linear system in matrix form.

(**c**) Find the augmented matrix.

31. Write the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} -2 & -1 & 0 & 4 & 5 \\ -3 & 2 & 7 & 8 & 3 \\ 1 & 0 & 0 & 2 & 4 \\ 3 & 0 & 1 & 3 & 6 \end{array}\right].$$

32. Write the following linear system in matrix form:

$$\begin{aligned} -2x_1 + 3x_2 &= 5 \\ x_1 - 5x_2 &= 4 \end{aligned}$$

33. Write the following linear system in matrix form:

$$\begin{aligned} 2x_1 + 3x_2 &= 0 \\ 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 &= 0 \end{aligned}$$

34. Write the linear system whose augmented matrix is

(**a**) $\left[\begin{array}{cccc|c} 2 & 1 & 3 & 4 & 0 \\ 3 & -1 & 2 & 0 & 3 \\ -2 & 1 & -4 & 3 & 2 \end{array}\right]$.

(**b**) $\left[\begin{array}{cccc|c} 2 & 1 & 3 & 4 & 0 \\ 3 & -1 & 2 & 0 & 3 \\ -2 & 1 & -4 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

35. How are the linear systems obtained in Exercise 34 related?

36. Write each of the following linear systems as a linear combination of the columns of the coefficient matrix:

(**a**) $\begin{aligned} 3x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 + 4x_3 &= -2 \end{aligned}$

(**b**) $\begin{aligned} -x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= -2 \\ 3x_1 + x_2 &= 1 \end{aligned}$

37. Write each of the following linear combinations of columns as a linear system of the form in (4):

(**a**) $x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

(**b**) $x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

38. Write each of the following as a linear system in matrix form:

(**a**) $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(**b**) $x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

39. Determine a solution to each of the following linear systems, using the fact that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A:

(**a**) $\begin{bmatrix} 1 & 2 & 1 \\ -3 & 6 & -3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(**b**) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$