1.6 Exercises

In Exercises 1 through 8, sketch **u** *and its image under each given matrix transformation f.*

1. $f: R^2 \to R^2$ defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

2. $f: R^2 \to R^2$ **(reflection with respect to the *y*-axis)** defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

3. $f: R^2 \to R^2$ is a counterclockwise rotation through $30°$; $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

4. $f: R^2 \to R^2$ is a counterclockwise rotation through $\frac{2}{3}\pi$ radians; $\mathbf{u} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

5. $f: R^2 \to R^2$ defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

6. $f: R^2 \to R^2$ defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

7. $f: R^2 \to R^3$ defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

8. $f: R^3 \to R^3$ defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

In Exercises 9 through 14, let $f: R^2 \to R^3$ be the matrix transformation defined by $f(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Determine whether each given vector $\mathbf{w}$ is in the range of f.

9. $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ **10.** $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ **11.** $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

12. $\mathbf{w} = \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix}$ **13.** $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ **14.** $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

In Exercises 15 through 17, give a geometric description of the matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{u}) = A\mathbf{u}$ for each given matrix A.

15. **(a)** $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

16. **(a)** $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

17. **(a)** $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

18. Some matrix transformations f have the property that $f(\mathbf{u}) = f(\mathbf{v})$, when $\mathbf{u} \neq \mathbf{v}$. That is, the images of different vectors can be the same. For each of the following matrix transformations $f: R^2 \to R^2$ defined by $f(\mathbf{u}) = A\mathbf{u}$, find two different vectors $\mathbf{u}$ and $\mathbf{v}$ such that $f(\mathbf{u}) = f(\mathbf{v}) = \mathbf{w}$ for the given vector $\mathbf{w}$.

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

19. Let $f: R^2 \to R^2$ be the linear transformation defined by $f(\mathbf{u}) = A\mathbf{u}$, where

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

For $\phi = 30°$, f defines a counterclockwise rotation by an angle of 30°.

(a) If $T_1(\mathbf{u}) = A^2\mathbf{u}$, describe the action of T_1 on $\mathbf{u}$.

(b) If $T_2(\mathbf{u}) = A^{-1}\mathbf{u}$, describe the action of T_2 on $\mathbf{u}$.

(c) What is the smallest positive value of k for which $T(\mathbf{u}) = A^k\mathbf{u} = \mathbf{u}$?

20. Let $f: R^n \to R^m$ be a matrix transformation defined by $f(\mathbf{u}) = A\mathbf{u}$, where A is an $m \times n$ matrix.

(a) Show that $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ for any $\mathbf{u}$ and $\mathbf{v}$ in R^n.

(b) Show that $f(c\mathbf{u}) = cf(\mathbf{u})$ for any $\mathbf{u}$ in R^n and any real number c.

(c) Show that $f(c\mathbf{u} + d\mathbf{v}) = cf(\mathbf{u}) + df(\mathbf{v})$ for any $\mathbf{u}$ and $\mathbf{v}$ in R^n and any real numbers c and d.

21. Let $f: R^n \to R^m$ be a matrix transformation defined by $f(\mathbf{u}) = A\mathbf{u}$, where A is an $m \times n$ matrix. Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n such that $f(\mathbf{u}) = \mathbf{0}$ and $f(\mathbf{v}) = \mathbf{0}$, where

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

then $f(c\mathbf{u} + d\mathbf{v}) = \mathbf{0}$ for any real numbers c and d.

22. **(a)** Let $O: R^n \to R^m$ be the matrix transformation defined by $O(\mathbf{u}) = O\mathbf{u}$, where O is the $m \times n$ zero matrix. Show that $O(\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u}$ in R^n.

(b) Let $I: R^n \to R^n$ be the matrix transformation defined by $I(\mathbf{u}) = I_n\mathbf{u}$, where I_n is the identity matrix. (See Section 1.5.) Show that $I(\mathbf{u}) = \mathbf{u}$ for all $\mathbf{u}$ in R^n.