Multiply the third row by $\left(\frac{2}{4+i}\right)$:

$$\left[\begin{array}{ccc|c} 1 & \frac{2}{i} & \frac{1-i}{i} & \frac{1-2i}{i} \\ 0 & 1 & \frac{2+i}{2i} & \frac{-2+i}{2i} \\ 0 & 0 & 1 & -1 \end{array}\right].$$

Add $\left(-\frac{2+i}{2i}\right)$ times the third row to the second row and $\left(-\frac{1-i}{i}\right)$ times the third row to the first row:

$$\left[\begin{array}{ccc|c} 1 & \frac{2}{i} & 0 & \frac{2-3i}{i} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

Add $\left(-\frac{2}{i}\right)$ times the second row to the first row:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

Hence the solution is $x = -3$, $y = 1$, $z = -1$. ■

Key Terms

Gaussian elimination
Gauss–Jordan reduction
Homogeneous system
Back substitution
Quadratic interpolation
Quadratic interpolant
Global positioning system
Chemical balance equations

2.2 Exercises

1. Each of the given linear systems is in row echelon form. Solve the system.

(a) $\begin{aligned} x + 2y - z &= 6 \\ y + z &= 5 \\ z &= 4 \end{aligned}$ **(b)** $\begin{aligned} x - 3y + 4z + w &= 0 \\ z - w &= 4 \\ w &= 1 \end{aligned}$

2. Each of the given linear systems is in row echelon form. Solve the system.

(a) $\begin{aligned} x + y - z + 2w &= 4 \\ w &= 5 \end{aligned}$ **(b)** $\begin{aligned} x - y + z &= 0 \\ y + 2z &= 0 \\ z &= 1 \end{aligned}$

3. Each of the given linear systems is in reduced row echelon form. Solve the system.

(a) $\begin{aligned} x + y \quad &= 2 \\ z + w &= -3 \end{aligned}$ **(b)** $\begin{aligned} x \quad &= 3 \\ y \quad &= 0 \\ z &= 1 \end{aligned}$

4. Each of the given linear systems is in reduced row echelon form. Solve the system.

(a) $\begin{aligned} x \quad - 2z &= 5 \\ y + z &= 2 \end{aligned}$ **(b)** $\begin{aligned} x \quad &= 1 \\ y \quad &= 2 \\ z - w &= 4 \end{aligned}$

5. Consider the linear system

$$\begin{aligned} x + y + 2z &= -1 \\ x - 2y + z &= -5 \\ 3x + y + z &= 3. \end{aligned}$$

(a) Find all solutions, if any exist, by using the Gaussian elimination method.

(b) Find all solutions, if any exist, by using the Gauss–Jordan reduction method.

6. Repeat Exercise 5 for each of the following linear systems:

(a)
$$\begin{aligned} x + y + 2z + 3w &= 13 \\ x - 2y + z + w &= 8 \\ 3x + y + z - w &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z &= 1 \\ x + y - 2z &= 3 \\ 2x + y + z &= 2 \end{aligned}$$

(c)
$$\begin{aligned} 2x + y + z - 2w &= 1 \\ 3x - 2y + z - 6w &= -2 \\ x + y - z - w &= -1 \\ 6x + z - 9w &= -2 \\ 5x - y + 2z - 8w &= 3 \end{aligned}$$

In Exercises 7 through 9, solve the linear system, with the given augmented matrix, if it is consistent.

7. (a) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 5 & 7 & 9 & 0 \end{array}\right]$ (d) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right]$

8. (a) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 8 \\ 1 & 3 & 0 & 1 & 7 \\ 1 & 0 & 2 & 1 & 3 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 1 & 1 & 3 & -3 & 0 \\ 0 & 2 & 1 & -3 & 3 \\ 1 & 0 & 2 & -1 & -1 \end{array}\right]$

9. (a) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 2 & 0 & 1 & 4 \\ 1 & 0 & 2 & 5 \\ 1 & 2 & 3 & 11 \\ 2 & 1 & 4 & 12 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right]$

10. Find a 2×1 matrix $\mathbf{x}$ with entries not all zero such that

$$A\mathbf{x} = 4\mathbf{x}, \quad \text{where } A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}.$$

[*Hint*: Rewrite the matrix equation $A\mathbf{x} = 4\mathbf{x}$ as $4\mathbf{x} - A\mathbf{x} = (4I_2 - A)\mathbf{x} = \mathbf{0}$, and solve the homogeneous linear system.]

11. Find a 2×1 matrix $\mathbf{x}$ with entries not all zero such that

$$A\mathbf{x} = 3\mathbf{x}, \quad \text{where } A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

12. Find a 3×1 matrix $\mathbf{x}$ with entries not all zero such that

$$A\mathbf{x} = 3\mathbf{x}, \quad \text{where } A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

13. Find a 3×1 matrix $\mathbf{x}$ with entries not all zero such that

$$A\mathbf{x} = 1\mathbf{x}, \quad \text{where } A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

14. In the following linear system, determine all values of a for which the resulting linear system has

(a) no solution;

(b) a unique solution;

(c) infinitely many solutions:

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

15. Repeat Exercise 14 for the linear system

$$\begin{aligned} x + y + z &= 2 \\ 2x + 3y + 2z &= 5 \\ 2x + 3y + (a^2 - 1)z &= a + 1. \end{aligned}$$

16. Repeat Exercise 14 for the linear system

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a. \end{aligned}$$

17. Repeat Exercise 14 for the linear system

$$\begin{aligned} x + y &= 3 \\ x + (a^2 - 8)y &= a. \end{aligned}$$

18. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Show that the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $ad - bc \neq 0$.

19. Show that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is row equivalent to I_2 if and only if $ad - bc \neq 0$.

20. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find x, y, z so that $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$.

21. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find x, y, z so that $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$.

22. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find an equation relating a, b, and c so that we can always compute values of x, y, and z for which

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

23. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find an equation relating a, b, and c so that we can always compute values of x, y, and z for which

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Exercises 24 and 25 are optional.

24. **(a)** Formulate the definitions of column echelon form and reduced column echelon form of a matrix.

(b) Prove that every $m \times n$ matrix is column equivalent to a matrix in column echelon form.

25. Prove that every $m \times n$ matrix is column equivalent to a unique matrix in reduced column echelon form.

26. Find an equation relating a, b, and c so that the linear system

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 3y + 3z &= b \\ 5x + 9y - 6z &= c \end{aligned}$$

is consistent for any values of a, b, and c that satisfy that equation.

27. Find an equation relating a, b, and c so that the linear system

$$\begin{aligned} 2x + 2y + 3z &= a \\ 3x - y + 5z &= b \\ x - 3y + 2z &= c \end{aligned}$$

is consistent for any values of a, b, and c that satisfy that equation.

28. Show that the homogeneous system

$$\begin{aligned} (a - r)x + \qquad dy &= 0 \\ cx + (b - r)y &= 0 \end{aligned}$$

has a nontrivial solution if and only if r satisfies the equation $(a - r)(b - r) - cd = 0$.

29. Let $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$, be a consistent linear system.

(a) Show that if $\mathbf{x}_p$ is a particular solution to the given nonhomogeneous system and $\mathbf{x}_h$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_p + \mathbf{x}_h$ is a solution to the given system $A\mathbf{x} = \mathbf{b}$.

(b) Show that every solution $\mathbf{x}$ to the nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution to the given nonhomogeneous system and $\mathbf{x}_h$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

[*Hint*: Let $\mathbf{x} = \mathbf{x}_p + (\mathbf{x} - \mathbf{x}_p)$.]

30. Determine the quadratic interpolant to each of the given data sets. Follow the procedure in Example 7.

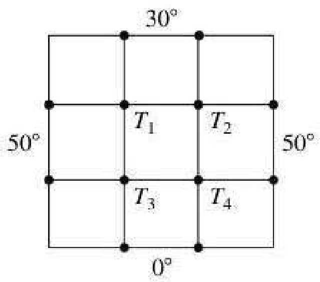
(a) $\{(0, 2), (1, 5), (2, 14)\}$

(b) $\{(-1, 2), (3, 14), (0, -1)\}$

31. ***(Calculus Required)*** Construct a linear system of equations to determine a quadratic polynomial $p(x) = ax^2 + bx + c$ that satisfies the conditions $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$, where $f(x) = e^{2x}$.

32. ***(Calculus Required)*** Construct a linear system of equations to determine a quadratic polynomial $p(x) = ax^2 + bx + c$ that satisfies the conditions $p(1) = f(1)$, $p'(1) = f'(1)$, and $p''(1) = f''(1)$, where $f(x) = xe^{x-1}$.

33. Determine the temperatures at the interior points T_i, $i = 1, 2, 3, 4$ for the plate shown in the figure. (See Example 8.)



34. Determine the planar location (x, y) of a GPS receiver, using coordinates of the centers and radii for the three circles given in the following tables:

(a)

Circle	*Center*	*Radius*
1	$(-15, 20)$	25
2	$(5, -12)$	13
3	$(9, 40)$	41

(b)

Circle	*Center*	*Radius*
1	$(-10, 13)$	25
2	$(10, -19)$	13
3	$(14, 33)$	41

35. The location of a GPS receiver in a two-dimensional system is $(-4, 3)$. The data used in the calculation are given in the table, except that the radius of the first circle is missing. Determine the value of the missing piece of data.

Circle	*Center*	*Radius*
1	$(-16, 38)$	?
2	$(7, -57)$	61
3	$(32, 80)$	85

36. The location of a GPS receiver in a two-dimensional system is $(6, 8)$. The data used in the calculation are given in the table, except that the radii of circles 1 and 2 are missing. Determine the values of missing pieces of data.

Circle	*Center*	*Radius*
1	$(3, 4)$	?
2	$(10, 5)$	?
3	$(18, 3)$	13

37. Suppose you have a "special edition" GPS receiver for two-dimensional systems that contains three special buttons, labeled C1, C2, and C3. Each button when depressed draws a circle that corresponds to data received from one of three closest satellites. You depress button C1 and then C2. The image on your handheld unit shows a pair of circles that are tangent to each other. What is the location of the GPS receiver? Explain.

38. Rust is formed when there is a chemical reaction between iron and oxygen. The compound that is formed is the reddish brown scales that cover the iron object. Rust is iron oxide whose chemical formula is Fe_2O_3. So a chemical equation for rust is

$$Fe + O_2 \rightarrow Fe_2O_3.$$

Balance this equation.

39. Ethane is a gas similar to methane that burns in oxygen to give carbon dioxide gas and steam. The steam condenses to form water droplets. The chemical equation for this reaction is

$$C_2H_6 + O_2 \rightarrow CO_2 + H_2O.$$

Balance this equation.

In Exercises 40 and 41, solve each given linear system.

40. $$\begin{aligned}(1-i)x + \quad (2+2i)y &= 1\\ (1+2i)x + (-2+2i)y &= i\end{aligned}$$

41. $$\begin{aligned}x + y \qquad &= 3 - i\\ ix + y + \; z &= 3\\ y + iz &= 3\end{aligned}$$

In Exercises 42 and 43, solve each linear system whose augmented matrix is given.

42. $$\left[\begin{array}{cc|c} 1-i & 2+2i & i \\ 1+i & -2+2i & -2 \end{array}\right]$$

43. $$\left[\begin{array}{ccc|c} 1 & i & -i & -2+2i \\ 2i & -i & 2 & -2 \\ 1 & 2 & 3i & 2i \end{array}\right]$$

44. Determine whether the software you are using has a command for computing the reduced row echelon form of a matrix. If it does, experiment with that command on some of the previous exercises.