Then

$$|A| = -2 \quad \text{and} \quad |B| = 5.$$

On the other hand, $AB = \begin{bmatrix} 4 & 3 \\ 10 & 5 \end{bmatrix}$, and $|AB| = -10 = |A||B|$. ■

**Corollary 3.2** If $A$ is nonsingular, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

***Proof***

Exercise 18. ■

**Corollary 3.3** If $A$ and $B$ are similar matrices, then $\det(A) = \det(B)$.

***Proof***

Exercise 33. ■

The determinant of a sum of two $n \times n$ matrices $A$ and $B$ is, in general, not the sum of the determinants of $A$ and $B$. The best result we can give along these lines is that if $A$, $B$, and $C$ are $n \times n$ matrices all of whose entries are equal except for the $k$th row (column), and the $k$th row (column) of $C$ is the sum of the $k$th rows (columns) of $A$ and $B$, then $\det(C) = \det(A) + \det(B)$. We shall not prove this result, but will consider an example.

**EXAMPLE 11** Let

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 4 \\ 1 & -2 & -4 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 4 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $|A| = 8$, $|B| = -9$, and $|C| = -1$, so $|C| = |A| + |B|$. ■

## Key Terms

Properties of the determinant
Elementary matrix
Reduction to triangular form

## 3.2 Exercises

**1.** Compute the following determinants via reduction to triangular form or by citing a particular theorem or corollary:

**(a)** $\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$ **(b)** $\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$

**(c)** $\begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ **(d)** $\begin{vmatrix} 4 & 1 & 3 \\ 2 & 3 & 0 \\ 1 & 3 & 2 \end{vmatrix}$

(e) $\begin{vmatrix} 4 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$ (f) $\begin{vmatrix} 4 & 2 & 3 & -4 \\ 3 & -2 & 1 & 5 \\ -2 & 0 & 1 & -3 \\ 8 & -2 & 6 & 4 \end{vmatrix}$

**2.** Compute the following determinants via reduction to triangular form or by citing a particular theorem or corollary:

(a) $\begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{vmatrix}$

(c) $\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{vmatrix}$ (d) $\begin{vmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

(e) $\begin{vmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 5 & 3 & 5 \end{vmatrix}$

(f) $\begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix}$

**3.** If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3$, find

$$\begin{vmatrix} a_1 + 2b_1 - 3c_1 & a_2 + 2b_2 - 3c_2 & a_3 + 2b_3 - 3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

**4.** If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -2$, find

$$\begin{vmatrix} a_1 - \frac{1}{2}a_3 & a_2 & a_3 \\ b_1 - \frac{1}{2}b_3 & b_2 & b_3 \\ c_1 - \frac{1}{2}c_3 & c_2 & c_3 \end{vmatrix}.$$

**5.** If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 4$, find

$$\begin{vmatrix} a_1 & a_2 & 4a_3 - 2a_2 \\ b_1 & b_2 & 4b_3 - 2b_2 \\ \frac{1}{2}c_1 & \frac{1}{2}c_2 & 2c_3 - c_2 \end{vmatrix}.$$

**6.** Verify that $\det(AB) = \det(A)\det(B)$ for the following:

**(a)** $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$

**(b)** $A = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 3 & 2 \\ 0 & 0 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 5 & 0 \\ 2 & 1 & -2 \end{bmatrix}$

**7.** Evaluate:

**(a)** $\begin{vmatrix} -4 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 0 & 3 \end{vmatrix}$

**(b)** $\begin{vmatrix} 2 & 0 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ 3 & 2 & 4 & 0 \\ 4 & 2 & 1 & -5 \end{vmatrix}$

**(c)** $\begin{vmatrix} t-1 & -1 & -2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{vmatrix}$

**(d)** $\begin{vmatrix} t+1 & 4 \\ 2 & t-3 \end{vmatrix}$

**8.** Is $\det(AB) = \det(BA)$? Justify your answer.

**9.** If $\det(AB) = 0$, is $\det(A) = 0$ or $\det(B) = 0$? Give reasons for your answer.

**10.** Show that if $k$ is a scalar and $A$ is $n \times n$, then $\det(kA) = k^n \det(A)$.

**11.** Show that if $A$ is $n \times n$ with $n$ odd and skew symmetric, then $\det(A) = 0$.

**12.** Show that if $A$ is a matrix such that in each row and in each column one and only one element is not equal to 0, then $\det(A) \neq 0$.

**13.** Show that $\det(AB^{-1}) = \dfrac{\det(A)}{\det(B)}$.

**14.** Show that if $AB = I_n$, then $\det(A) \neq 0$ and $\det(B) \neq 0$.

**15.** **(a)** Show that if $A = A^{-1}$, then $\det(A) = \pm 1$.

**(b)** If $A^T = A^{-1}$, what is $\det(A)$?

**16.** Show that if $A$ and $B$ are square matrices, then $\det\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}\right) = (\det A)(\det B)$.

**17.** If $A$ is a nonsingular matrix such that $A^2 = A$, what is $\det(A)$?

**18.** Prove Corollary 3.2.

**19.** Show that if $A$, $B$, and $C$ are square matrices, then $\det\left(\begin{bmatrix} A & O \\ C & B \end{bmatrix}\right) = (\det A)(\det B)$.

**20.** Show that if $A$ and $B$ are both $n \times n$, then

(**a**) $\det(A^T B^T) = \det(A)\det(B^T)$;

(**b**) $\det(A^T B^T) = \det(A^T)\det(B)$.

**21.** Verify the result in Exercise 16 for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$.

**22.** Use the properties of Section 3.2 to prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

(*Hint*: Use factorization.) This determinant is called a **Vandermonde* determinant**.

**23.** If $\det(A) = 2$, find $\det(A^5)$.

**24.** Use Theorem 3.8 to determine which of the following matrices are nonsingular:

(**a**) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix}$ (**b**) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

**25.** Use Theorem 3.8 to determine which of the following matrices are nonsingular:

(**a**) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 1 & -7 & 2 \end{bmatrix}$

(**b**) $\begin{bmatrix} 1 & 2 & 0 & 5 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 2 & 0 \\ 0 & 1 & 2 & -7 \end{bmatrix}$

**26.** Use Theorem 3.8 to determine all values of $t$ so that the following matrices are nonsingular:

(**a**) $\begin{bmatrix} t & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ (**b**) $\begin{bmatrix} t & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & t \end{bmatrix}$

(**c**) $\begin{bmatrix} t & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 1 & 0 & 0 & t \end{bmatrix}$

**27.** Use Corollary 3.1 to find out whether the following homogeneous system has a nontrivial solution (do *not* solve):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

**28.** Repeat Exercise 27 for the following homogeneous system:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

**29.** Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be an upper triangular matrix. Prove that $A$ is nonsingular if and only if $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$.

**30.** Let $A$ be a $3 \times 3$ matrix with $\det(A) = 3$.

(**a**) What is the reduced row echelon form to which $A$ is row equivalent?

(**b**) How many solutions does the homogeneous system $A\mathbf{x} = \mathbf{0}$ have?

**31.** Let $A$ be a $4 \times 4$ matrix with $\det(A) = 0$.

(**a**) Describe the reduced row echelon form matrix to which $A$ is row equivalent.

(**b**) How many solutions does the homogeneous system $A\mathbf{x} = \mathbf{0}$ have?

**32.** Let $A^2 = A$. Prove that either $A$ is singular or $\det(A) = 1$.

**33.** Prove Corollary 3.3.

**34.** Let $AB = AC$. Prove that if $\det(A) \neq 0$, then $B = C$.

**35.** Determine whether the software you are using has a command for computing the determinant of a matrix. If it does, verify the computations in Examples 8, 10, and 11. Experiment further by finding the determinant of the matrices in Exercises 1 and 2.

**36.** Assuming that your software has a command to compute the determinant of a matrix, read the accompanying software documentation to determine the method used. Is the description closest to that in Section 3.1, Example 8 in Section 3.2, or the material in Section 2.5?

*Alexandre-Théophile Vandermonde (1735–1796) was born and died in Paris. His father was a physician who encouraged his son to pursue a career in music. Vandermonde followed his father's advice and did not get interested in mathematics until he was 35 years old. His entire mathematical output consisted of four papers. He also published papers on chemistry and on the manufacture of steel. Although Vandermonde is best known for his determinant, it does not appear in any of his four papers. It is believed that someone mistakenly attributed this determinant to him. However, in his fourth mathematical paper, Vandermonde made significant contributions to the theory of determinants. He was a staunch republican who fully backed the French Revolution.