3.3 Exercises

1. Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{bmatrix}$. Find the following minors:

(a) $\det(M_{13})$ **(b)** $\det(M_{22})$

(c) $\det(M_{31})$ **(d)** $\det(M_{32})$

2. Let $A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & -2 & 4 \\ -1 & 1 & -3 & -2 \\ 0 & 2 & -1 & 5 \end{bmatrix}$. Find the following minors:

(a) $\det(M_{12})$ **(b)** $\det(M_{23})$

(c) $\det(M_{33})$ **(d)** $\det(M_{41})$

3. Let $A = \begin{bmatrix} -1 & 2 & 3 \\ -2 & 5 & 4 \\ 0 & 1 & -3 \end{bmatrix}$. Find the following cofactors:

(a) A_{13} **(b)** A_{21} **(c)** A_{32} **(d)** A_{33}

4. Let $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & -4 & -1 \\ 3 & 2 & 4 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix}$. Find the following cofactors:

(a) A_{12} **(b)** A_{23} **(c)** A_{33} **(d)** A_{41}

5. Use Theorem 3.10 to evaluate the determinants in Exercise 1(a), (d), and (e) of Section 3.2.

6. Use Theorem 3.10 to evaluate the determinants in Exercise 1(b), (c), and (f) of Section 3.2.

7. Use Theorem 3.10 to evaluate the determinants in Exercise 2(a), (c), and (f) of Section 3.2.

8. Use Theorem 3.10 to evaluate the determinants in Exercise 2(b), (d), and (e) of Section 3.2.

9. Show by a column (row) expansion that if $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is upper (lower) triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

10. If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is a 3×3 matrix, develop the general expression for $\det(A)$ by expanding

(a) along the second column;

(b) along the third row.

Compare these answers with those obtained for Example 8 in Section 3.1.

11. Find all values of t for which

(a) $\begin{vmatrix} t-2 & 2 \\ 3 & t-3 \end{vmatrix} = 0;$

(b) $\begin{vmatrix} t-1 & -4 \\ 0 & t-4 \end{vmatrix} = 0.$

12. Find all values of t for which

$$\begin{vmatrix} t-1 & 0 & 1 \\ -2 & t+2 & -1 \\ 0 & 0 & t+1 \end{vmatrix} = 0.$$

13. Let A be an $n \times n$ matrix.

(a) Show that $f(t) = \det(tI_n - A)$ is a polynomial in t of degree n.

(b) What is the coefficient of t^n in $f(t)$?

(c) What is the constant term in $f(t)$?

14. Verify your answers to Exercise 13 with the following matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

15. Let T be the triangle with vertices $(3, 3)$, $(-1, -1)$, $(4, 1)$.

(a) Find the area of the triangle T.

(b) Find the coordinates of the vertices of the image of T under the matrix transformation with matrix representation

$$A = \begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}.$$

(c) Find the area of the triangle whose vertices are obtained in part (b).

16. Find the area of the parallelogram with vertices $(2, 3)$, $(5, 3)$, $(4, 5)$, $(7, 5)$.

17. Let Q be the quadrilateral with vertices $(-2, 3)$, $(1, 4)$, $(3, 0)$, and $(-1, -3)$. Find the area of Q.

18. Prove that a rotation leaves the area of a triangle unchanged.

19. Let T be the triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3), and let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let f be the matrix transformation defined by $f(\mathbf{v}) = A\mathbf{v}$ for a vector $\mathbf{v}$ in R^2. First, compute the vertices of $f(T)$ and the image of T under f, and then show that the area of $f(T)$ is $|\det(A)| \cdot$ area of T.