CHAPTER 6 APPLICATIONS OF DEFINITE INTEGRALS

6.1 VOLUMES BY SLICING AND ROTATION ABOUT AN AXIS

1. (a) $A = \pi(\text{radius})^2$ and radius $= \sqrt{1-x^2} \Rightarrow A(x) = \pi(1-x^2)$

 (b) $A = \text{width} \cdot \text{height}$, width $=$ height $= 2\sqrt{1-x^2} \Rightarrow A(x) = 4(1-x^2)$

 (c) $A = (\text{side})^2$ and diagonal $= \sqrt{2}(\text{side}) \Rightarrow A = \frac{(\text{diagonal})^2}{2}$; diagonal $= 2\sqrt{1-x^2} \Rightarrow A(x) = 2(1-x^2)$

 (d) $A = \frac{\sqrt{3}}{4}(\text{side})^2$ and side $= 2\sqrt{1-x^2} \Rightarrow A(x) = \sqrt{3}(1-x^2)$

2. (a) $A = \pi(\text{radius})^2$ and radius $= \sqrt{x} \Rightarrow A(x) = \pi x$

 (b) $A = \text{width} \cdot \text{height}$, width $=$ height $= 2\sqrt{x} \Rightarrow A(x) = 4x$

 (c) $A = (\text{side})^2$ and diagonal $= \sqrt{2}(\text{side}) \Rightarrow A = \frac{(\text{diagonal})^2}{2}$; diagonal $= 2\sqrt{x} \Rightarrow A(x) = 2x$

 (d) $A = \frac{\sqrt{3}}{4}(\text{side})^2$ and side $= 2\sqrt{x} \Rightarrow A(x) = \sqrt{3}x$

3. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left(\sqrt{x} - (-\sqrt{x})\right)^2}{2} = 2x$ (see Exercise 1c); $a = 0$, $b = 4$;

 $V = \int_a^b A(x)\,dx = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16$

4. $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi[(2-x^2)-x^2]^2}{4} = \frac{\pi[2(1-x^2)]^2}{4} = \pi(1-2x^2+x^4)$; $a = -1$, $b = 1$;

 $V = \int_a^b A(x)\,dx = \int_{-1}^1 \pi(1-2x^2+x^4)\,dx = \pi\left[x - \frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16\pi}{15}$

5. $A(x) = (\text{edge})^2 = \left[\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right]^2 = \left(2\sqrt{1-x^2}\right)^2 = 4(1-x^2)$; $a = -1$, $b = 1$;

 $V = \int_a^b A(x)\,dx = \int_{-1}^1 4(1-x^2)\,dx = 4\left[x - \frac{x^3}{3}\right]_{-1}^1 = 8\left(1 - \frac{1}{3}\right) = \frac{16}{3}$

6. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left[\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right]^2}{2} = \frac{\left(2\sqrt{1-x^2}\right)^2}{2} = 2(1-x^2)$ (see Exercise 1c); $a = -1$, $b = 1$;

 $V = \int_a^b A(x)\,dx = 2\int_{-1}^1 (1-x^2)\,dx = 2\left[x - \frac{x^3}{3}\right]_{-1}^1 = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$

7. (a) STEP 1) $A(x) = \frac{1}{2}(\text{side}) \cdot (\text{side}) \cdot \left(\sin\frac{\pi}{3}\right) = \frac{1}{2} \cdot \left(2\sqrt{\sin x}\right) \cdot \left(2\sqrt{\sin x}\right)\left(\sin\frac{\pi}{3}\right) = \sqrt{3}\sin x$

 STEP 2) $a = 0$, $b = \pi$

 STEP 3) $V = \int_a^b A(x)\,dx = \sqrt{3}\int_0^\pi \sin x\,dx = \left[-\sqrt{3}\cos x\right]_0^\pi = \sqrt{3}(1+1) = 2\sqrt{3}$

 (b) STEP 1) $A(x) = (\text{side})^2 = \left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right) = 4\sin x$

 STEP 2) $a = 0$, $b = \pi$

 STEP 3) $V = \int_a^b A(x)\,dx = \int_0^\pi 4\sin x\,dx = \left[-4\cos x\right]_0^\pi = 8$

8. (a) STEP 1) $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi}{4}(\sec x - \tan x)^2 = \frac{\pi}{4}(\sec^2 x + \tan^2 x - 2\sec x\tan x)$

 $= \frac{\pi}{4}\left[\sec^2 x + (\sec^2 x - 1) - 2\,\frac{\sin x}{\cos^2 x}\right]$

 STEP 2) $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$

 STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \frac{\pi}{4}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = \frac{\pi}{4}\left[2\tan x - x + 2\left(-\frac{1}{\cos x}\right)\right]_{-\pi/3}^{\pi/3}$

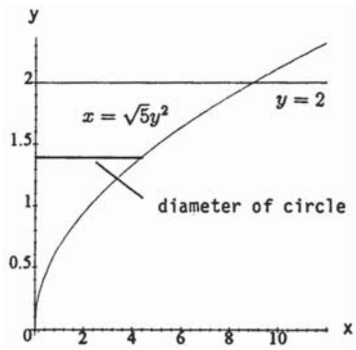
$= \frac{\pi}{4}\left[2\sqrt{3} - \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right) - \left(-2\sqrt{3} + \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right)\right)\right] = \frac{\pi}{4}\left(4\sqrt{3} - \frac{2\pi}{3}\right)$

(b) STEP 1) $A(x) = (\text{edge})^2 = (\sec x - \tan x)^2 = \left(2\sec^2 x - 1 - 2\,\frac{\sin x}{\cos^2 x}\right)$

STEP 2) $a = -\frac{\pi}{3},\ b = \frac{\pi}{3}$

STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = 2\left(2\sqrt{3} - \frac{\pi}{3}\right) = 4\sqrt{3} - \frac{2\pi}{3}$

9. $A(y) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{5}y^2 - 0\right)^2 = \frac{5\pi}{4}y^4$;
$c = 0,\ d = 2;\ V = \int_c^d A(y)\,dy = \int_0^2 \frac{5\pi}{4}y^4\,dy$
$= \left[\left(\frac{5\pi}{4}\right)\left(\frac{y^5}{5}\right)\right]_0^2 = \frac{\pi}{4}\left(2^5 - 0\right) = 8\pi$



10. $A(y) = \frac{1}{2}(\text{leg})(\text{leg}) = \frac{1}{2}\left[\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right]^2 = \frac{1}{2}\left(2\sqrt{1-y^2}\right)^2 = 2\left(1-y^2\right);\ c = -1,\ d = 1$;
$V = \int_c^d A(y)\,dy = \int_{-1}^1 2(1-y^2)\,dy = 2\left[y - \frac{y^3}{3}\right]_{-1}^1 = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$

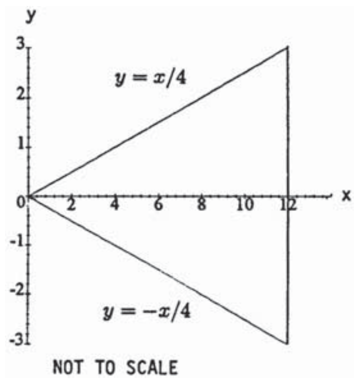
11. (a) It follows from Cavalieri's Principle that the volume of a column is the same as the volume of a right prism with a square base of side length s and altitude h. Thus, STEP 1) $A(x) = (\text{side length})^2 = s^2$; STEP 2) $a = 0,\ b = h$; STEP 3) $V = \int_a^b A(x)\,dx = \int_0^h s^2\,dx = s^2 h$

(b) From Cavalieri's Principle we conclude that the volume of the column is the same as the volume of the prism described above, regardless of the number of turns $\Rightarrow V = s^2 h$

12. 1) The solid and the cone have the same altitude of 12.

2) The cross sections of the solid are disks of diameter $x - \left(\frac{x}{2}\right) = \frac{x}{2}$. If we place the vertex of the cone at the origin of the coordinate system and make its axis of symmetry coincide with the x-axis then the cone's cross sections will be circular disks of diameter $\frac{x}{4} - \left(-\frac{x}{4}\right) = \frac{x}{2}$ (see accompanying figure).

3) The solid and the cone have equal altitudes and identical parallel cross sections. From Cavalieri's Principle we conclude that the solid and the cone have the same volume.



13. $R(x) = y = 1 - \frac{x}{2} \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2\left(1 - \frac{x}{2}\right)^2 dx = \pi\int_0^2\left(1 - x + \frac{x^2}{4}\right)dx = \pi\left[x - \frac{x^2}{2} + \frac{x^3}{12}\right]_0^2$
$= \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$

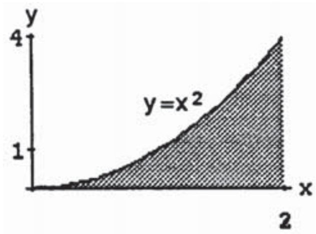
14. $R(y) = x = \frac{3y}{2} \Rightarrow V = \int_0^2 \pi[R(y)]^2\,dy = \pi\int_0^2\left(\frac{3y}{2}\right)^2 dy = \pi\int_0^2 \frac{9}{4}y^2\,dy = \pi\left[\frac{3}{4}y^3\right]_0^2 = \pi\cdot\frac{3}{4}\cdot 8 = 6\pi$

15. $R(x) = \tan\left(\frac{\pi}{4}y\right);\ u = \frac{\pi}{4}y \Rightarrow du = \frac{\pi}{4}dy \Rightarrow 4\,du = \pi\,dy;\ y = 0 \Rightarrow u = 0,\ y = 1 \Rightarrow u = \frac{\pi}{4}$;
$V = \int_0^1 \pi[R(y)]^2\,dy = \pi\int_0^1\left[\tan\left(\frac{\pi}{4}y\right)\right]^2 dy = 4\int_0^{\pi/4}\tan^2 u\,du = 4\int_0^{\pi/4}\left(-1 + \sec^2 u\right)du = 4[-u + \tan u]_0^{\pi/4}$

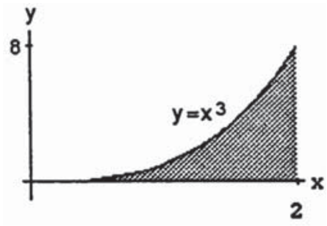
$= 4\left(-\frac{\pi}{4} + 1 - 0\right) = 4 - \pi$

16. $R(x) = \sin x \cos x$; $R(x) = 0 \Rightarrow a = 0$ and $b = \frac{\pi}{2}$ are the limits of integration; $V = \int_0^{\pi/2} \pi[R(x)]^2\, dx$
$= \pi \int_0^{\pi/2} (\sin x \cos x)^2\, dx = \pi \int_0^{\pi/2} \frac{(\sin 2x)^2}{4}\, dx$; $\left[u = 2x \Rightarrow du = 2\, dx \Rightarrow \frac{du}{8} = \frac{dx}{4}; x = 0 \Rightarrow u = 0,\right.$
$\left. x = \frac{\pi}{2} \Rightarrow u = \pi\right] \rightarrow V = \pi \int_0^{\pi} \frac{1}{8} \sin^2 u\, du = \frac{\pi}{8}\left[\frac{u}{2} - \frac{1}{4}\sin 2u\right]_0^{\pi} = \frac{\pi}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi^2}{16}$

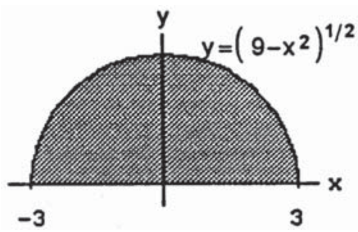
17. $R(x) = x^2 \Rightarrow V = \int_0^2 \pi[R(x)]^2\, dx = \pi \int_0^2 (x^2)^2\, dx$
$= \pi \int_0^2 x^4\, dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$



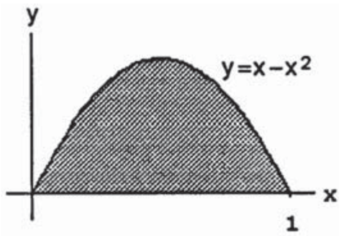
18. $R(x) = x^3 \Rightarrow V = \int_0^2 \pi[R(x)]^2\, dx = \pi \int_0^2 (x^3)^2\, dx$
$= \pi \int_0^2 x^6\, dx = \pi \left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$



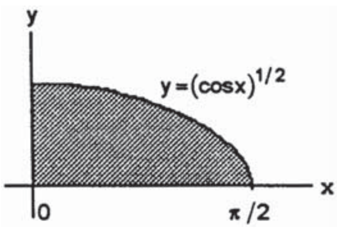
19. $R(x) = \sqrt{9 - x^2} \Rightarrow V = \int_{-3}^3 \pi[R(x)]^2\, dx = \pi \int_{-3}^3 (9 - x^2)\, dx$
$= \pi \left[9x - \frac{x^3}{3}\right]_{-3}^3 = 2\pi\left[9(3) - \frac{27}{3}\right] = 2 \cdot \pi \cdot 18 = 36\pi$



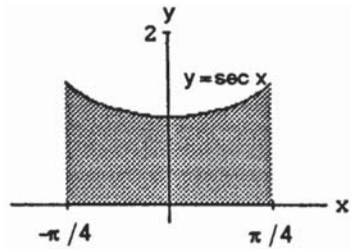
20. $R(x) = x - x^2 \Rightarrow V = \int_0^1 \pi[R(x)]^2\, dx = \pi \int_0^1 (x - x^2)^2\, dx$
$= \pi \int_0^1 (x^2 - 2x^3 + x^4)\, dx = \pi \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1$
$= \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{30}(10 - 15 + 6) = \frac{\pi}{30}$



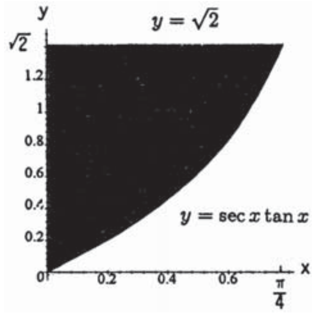
21. $R(x) = \sqrt{\cos x} \Rightarrow V = \int_0^{\pi/2} \pi[R(x)]^2\, dx = \pi \int_0^{\pi/2} \cos x\, dx$
$= \pi\left[\sin x\right]_0^{\pi/2} = \pi(1 - 0) = \pi$



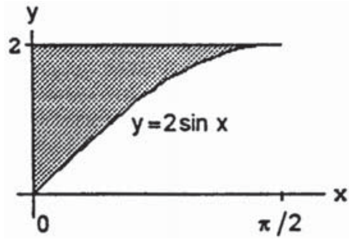
22. $R(x) = \sec x \Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi[R(x)]^2\,dx = \pi \int_{-\pi/4}^{\pi/4} \sec^2 x\,dx$
$= \pi\,[\tan x]_{-\pi/4}^{\pi/4} = \pi[1-(-1)] = 2\pi$



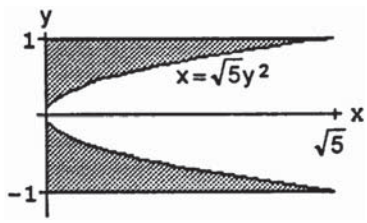
23. $R(x) = \sqrt{2} - \sec x \tan x \Rightarrow V = \int_0^{\pi/4} \pi[R(x)]^2\,dx$
$= \pi \int_0^{\pi/4} \left(\sqrt{2} - \sec x \tan x\right)^2 dx$
$= \pi \int_0^{\pi/4} \left(2 - 2\sqrt{2}\sec x \tan x + \sec^2 x \tan^2 x\right) dx$
$= \pi \left(\int_0^{\pi/4} 2\,dx - 2\sqrt{2}\int_0^{\pi/4} \sec x \tan x\,dx + \int_0^{\pi/4} (\tan x)^2 \sec^2 x\,dx\right)$
$= \pi \left([2x]_0^{\pi/4} - 2\sqrt{2}\,[\sec x]_0^{\pi/4} + \left[\frac{\tan^3 x}{3}\right]_0^{\pi/4}\right)$
$= \pi \left[\left(\frac{\pi}{2} - 0\right) - 2\sqrt{2}\left(\sqrt{2}-1\right) + \frac{1}{3}\left(1^3 - 0\right)\right] = \pi\left(\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3}\right)$



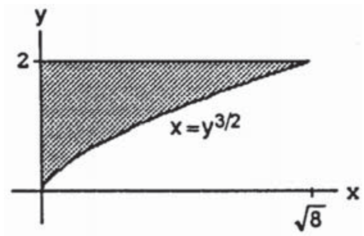
24. $R(x) = 2 - 2\sin x = 2(1-\sin x) \Rightarrow V = \int_0^{\pi/2} \pi[R(x)]^2\,dx$
$= \pi \int_0^{\pi/2} 4(1-\sin x)^2\,dx = 4\pi \int_0^{\pi/2} (1 + \sin^2 x - 2\sin x)\,dx$
$= 4\pi \int_0^{\pi/2} \left[1 + \frac{1}{2}(1 - \cos 2x) - 2\sin x\right] dx$
$= 4\pi \int_0^{\pi/2} \left(\frac{3}{2} - \frac{\cos 2x}{2} - 2\sin x\right)$
$= 4\pi \left[\frac{3}{2}x - \frac{\sin 2x}{4} + 2\cos x\right]_0^{\pi/2}$
$= 4\pi\left[\left(\frac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 2)\right] = \pi(3\pi - 8)$



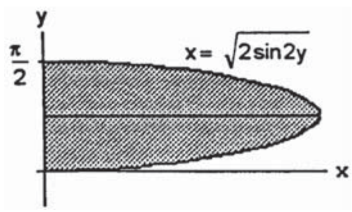
25. $R(y) = \sqrt{5} \cdot y^2 \Rightarrow V = \int_{-1}^{1} \pi[R(y)]^2\,dy = \pi \int_{-1}^{1} 5y^4\,dy$
$= \pi\,[y^5]_{-1}^{1} = \pi[1-(-1)] = 2\pi$



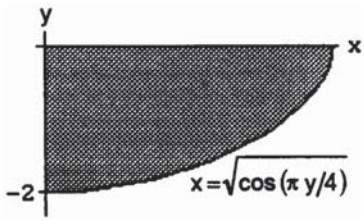
26. $R(y) = y^{3/2} \Rightarrow V = \int_0^2 \pi[R(y)]^2\,dy = \pi \int_0^2 y^3\,dy$
$= \pi\left[\frac{y^4}{4}\right]_0^2 = 4\pi$



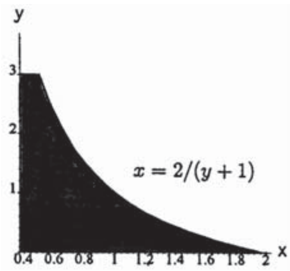
27. $R(y) = \sqrt{2\sin 2y} \Rightarrow V = \int_0^{\pi/2} \pi[R(y)]^2\,dy$
$= \pi \int_0^{\pi/2} 2\sin 2y\,dy = \pi\,[-\cos 2y]_0^{\pi/2}$
$= \pi[1-(-1)] = 2\pi$



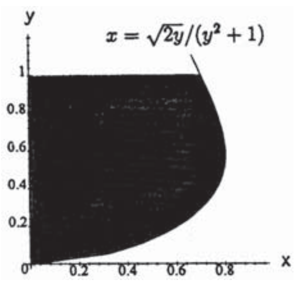
28. $R(y) = \sqrt{\cos \frac{\pi y}{4}} \Rightarrow V = \int_{-2}^{0} \pi[R(y)]^2\, dy$
$= \pi \int_{-2}^{0} \cos\left(\frac{\pi y}{4}\right) dy = 4\left[\sin \frac{\pi y}{4}\right]_{-2}^{0} = 4[0 - (-1)] = 4$



29. $R(y) = \frac{2}{y+1} \Rightarrow V = \int_0^3 \pi[R(y)]^2\, dy = 4\pi \int_0^3 \frac{1}{(y+1)^2}\, dy$
$= 4\pi \left[\frac{-1}{y+1}\right]_0^3 = 4\pi\left[-\frac{1}{4} - (-1)\right] = 3\pi$



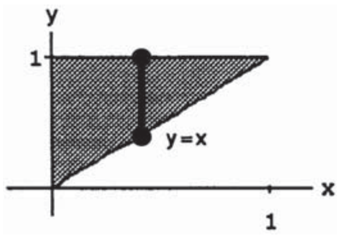
30. $R(y) = \frac{\sqrt{2y}}{y^2+1} \Rightarrow V = \int_0^1 \pi[R(y)]^2\, dy = \pi\int_0^1 2y\left(y^2+1\right)^{-2} dy;$
$[u = y^2 + 1 \Rightarrow du = 2y\, dy;\ y = 0 \Rightarrow u = 1,\ y = 1 \Rightarrow u = 2]$
$\rightarrow V = \pi\int_1^2 u^{-2}\, du = \pi\left[-\frac{1}{u}\right]_1^2 = \pi\left[-\frac{1}{2} - (-1)\right] = \frac{\pi}{2}$



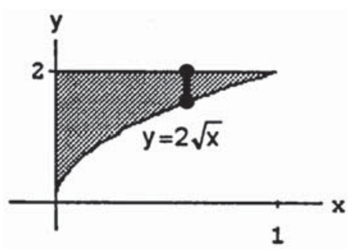
31. For the sketch given, $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$; $R(x) = 1$, $r(x) = \sqrt{\cos x}$; $V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$
$= \int_{-\pi/2}^{\pi/2} \pi(1 - \cos x)\, dx = 2\pi\int_0^{\pi/2}(1 - \cos x)\, dx = 2\pi[x - \sin x]_0^{\pi/2} = 2\pi\left(\frac{\pi}{2} - 1\right) = \pi^2 - 2\pi$

32. For the sketch given, $c = 0$, $d = \frac{\pi}{4}$; $R(y) = 1$, $r(y) = \tan y$; $V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy$
$= \pi\int_0^{\pi/4}\left(1 - \tan^2 y\right) dy = \pi\int_0^{\pi/4}\left(2 - \sec^2 y\right) dy = \pi[2y - \tan y]_0^{\pi/4} = \pi\left(\frac{\pi}{2} - 1\right) = \frac{\pi^2}{2} - \pi$

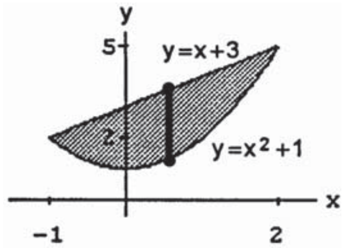
33. $r(x) = x$ and $R(x) = 1 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$
$= \int_0^1 \pi\left(1 - x^2\right) dx = \pi\left[x - \frac{x^3}{3}\right]_0^1 = \pi\left[\left(1 - \frac{1}{3}\right) - 0\right] = \frac{2\pi}{3}$



34. $r(x) = 2\sqrt{x}$ and $R(x) = 2 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$
$= \pi\int_0^1 (4 - 4x)\, dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1 = 4\pi\left(1 - \frac{1}{2}\right) = 2\pi$



35. $r(x) = x^2 + 1$ and $R(x) = x + 3$



$\Rightarrow V = \int_{-1}^{2} \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{-1}^{2} \left[(x+3)^2 - \left(x^2+1\right)^2\right] dx$

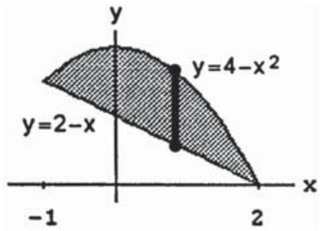
$= \pi \int_{-1}^{2} \left[(x^2 + 6x + 9) - (x^4 + 2x^2 + 1)\right] dx$

$= \pi \int_{-1}^{2} \left(-x^4 - x^2 + 6x + 8\right) dx$

$= \pi \left[-\frac{x^5}{5} - \frac{x^3}{3} + \frac{6x^2}{2} + 8x\right]_{-1}^{2}$

$= \pi \left[\left(-\frac{32}{5} - \frac{8}{3} + \frac{24}{2} + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + \frac{6}{2} - 8\right)\right] = \pi \left(-\frac{33}{5} - 3 + 28 - 3 + 8\right) = \pi \left(\frac{5 \cdot 30 - 33}{5}\right) = \frac{117\pi}{5}$

36. $r(x) = 2 - x$ and $R(x) = 4 - x^2$



$\Rightarrow V = \int_{-1}^{2} \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{-1}^{2} \left[\left(4 - x^2\right)^2 - (2-x)^2\right] dx$

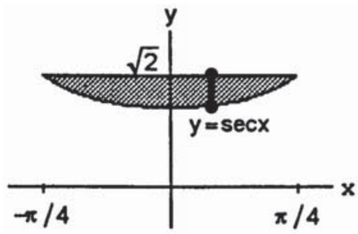
$= \pi \int_{-1}^{2} \left[(16 - 8x^2 + x^4) - (4 - 4x + x^2)\right] dx$

$= \pi \int_{-1}^{2} \left(12 + 4x - 9x^2 + x^4\right) dx$

$= \pi \left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-1}^{2}$

$= \pi \left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right] = \pi \left(15 + \frac{33}{5}\right) = \frac{108\pi}{5}$

37. $r(x) = \sec x$ and $R(x) = \sqrt{2}$

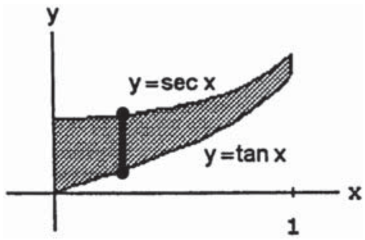


$\Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{-\pi/4}^{\pi/4} \left(2 - \sec^2 x\right) dx = \pi [2x - \tan x]_{-\pi/4}^{\pi/4}$

$= \pi \left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right] = \pi(\pi - 2)$

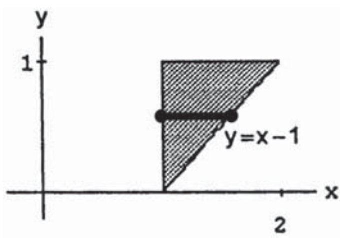
38. $R(x) = \sec x$ and $r(x) = \tan x$



$\Rightarrow V = \int_{0}^{1} \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{0}^{1} \left(\sec^2 x - \tan^2 x\right) dx = \pi \int_{0}^{1} 1\, dx = \pi [x]_0^1 = \pi$

39. $r(y) = 1$ and $R(y) = 1 + y$

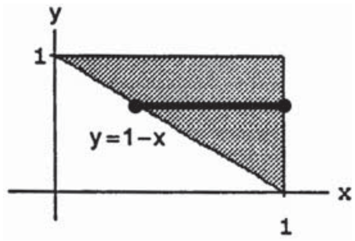


$\Rightarrow V = \int_{0}^{1} \pi \left([R(y)]^2 - [r(y)]^2\right) dy$

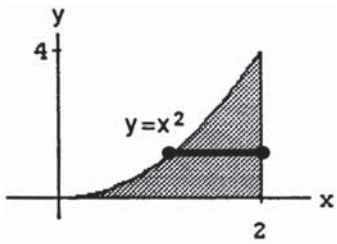
$= \pi \int_{0}^{1} \left[(1+y)^2 - 1\right] dy = \pi \int_{0}^{1} \left(1 + 2y + y^2 - 1\right) dy$

$= \pi \int_{0}^{1} \left(2y + y^2\right) dy = \pi \left[y^2 + \frac{y^3}{3}\right]_{0}^{1} = \pi \left(1 + \frac{1}{3}\right) = \frac{4\pi}{3}$

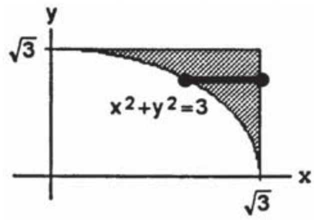
40. $R(y) = 1$ and $r(y) = 1 - y \Rightarrow V = \int_0^1 \pi \left([R(y)]^2 - [r(y)]^2\right) dy$
$= \pi \int_0^1 [1 - (1-y)^2]\, dy = \pi \int_0^1 [1 - (1 - 2y + y^2)]\, dy$
$= \pi \int_0^1 (2y - y^2)\, dy = \pi \left[y^2 - \frac{y^3}{3}\right]_0^1 = \pi\left(1 - \frac{1}{3}\right) = \frac{2\pi}{3}$



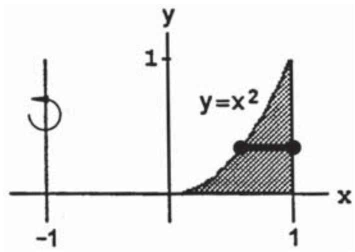
41. $R(y) = 2$ and $r(y) = \sqrt{y}$
$\Rightarrow V = \int_0^4 \pi \left([R(y)]^2 - [r(y)]^2\right) dy$
$= \pi \int_0^4 (4 - y)\, dy = \pi \left[4y - \frac{y^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$



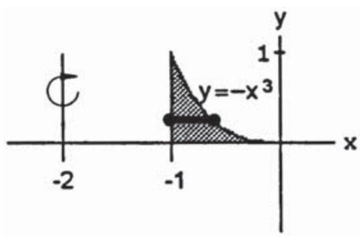
42. $R(y) = \sqrt{3}$ and $r(y) = \sqrt{3 - y^2}$
$\Rightarrow V = \int_0^{\sqrt{3}} \pi \left([R(y)]^2 - [r(y)]^2\right) dy$
$= \pi \int_0^{\sqrt{3}} [3 - (3 - y^2)]\, dy = \pi \int_0^{\sqrt{3}} y^2\, dy$
$= \pi \left[\frac{y^3}{3}\right]_0^{\sqrt{3}} = \pi\sqrt{3}$



43. $R(y) = 2$ and $r(y) = 1 + \sqrt{y}$
$\Rightarrow V = \int_0^1 \pi \left([R(y)]^2 - [r(y)]^2\right) dy$
$= \pi \int_0^1 \left[4 - \left(1 + \sqrt{y}\right)^2\right] dy$
$= \pi \int_0^1 \left(4 - 1 - 2\sqrt{y} - y\right) dy$
$= \pi \int_0^1 \left(3 - 2\sqrt{y} - y\right) dy$
$= \pi \left[3y - \frac{4}{3} y^{3/2} - \frac{y^2}{2}\right]_0^1$
$= \pi\left(3 - \frac{4}{3} - \frac{1}{2}\right) = \pi\left(\frac{18-8-3}{6}\right) = \frac{7\pi}{6}$



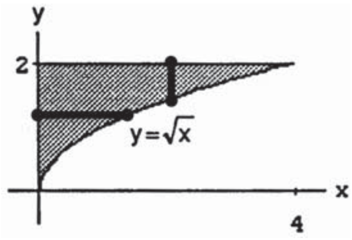
44. $R(y) = 2 - y^{1/3}$ and $r(y) = 1$
$\Rightarrow V = \int_0^1 \pi \left([R(y)]^2 - [r(y)]^2\right) dy$
$= \pi \int_0^1 \left[\left(2 - y^{1/3}\right)^2 - 1\right] dy$
$= \pi \int_0^1 \left(4 - 4y^{1/3} + y^{2/3} - 1\right) dy$
$= \pi \int_0^1 \left(3 - 4y^{1/3} + y^{2/3}\right) dy$
$= \pi \left[3y - 3y^{4/3} + \frac{3y^{5/3}}{5}\right]_0^1 = \pi\left(3 - 3 + \frac{3}{5}\right) = \frac{3\pi}{5}$



45. (a) $r(x) = \sqrt{x}$ and $R(x) = 2$

$\Rightarrow V = \int_0^4 \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_0^4 (4 - x)\, dx = \pi \left[4x - \frac{x^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$



(b) $r(y) = 0$ and $R(y) = y^2$

$\Rightarrow V = \int_0^2 \pi \left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 y^4\, dy = \pi \left[\frac{y^5}{5}\right]_0^2 = \frac{32\pi}{5}$

(c) $r(x) = 0$ and $R(x) = 2 - \sqrt{x} \Rightarrow V = \int_0^4 \pi \left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_0^4 \left(2 - \sqrt{x}\right)^2 dx$

$= \pi \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx = \pi \left[4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2}\right]_0^4 = \pi \left(16 - \frac{64}{3} + \frac{16}{2}\right) = \frac{8\pi}{3}$

(d) $r(y) = 4 - y^2$ and $R(y) = 4 \Rightarrow V = \int_0^2 \pi \left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[16 - \left(4 - y^2\right)^2\right] dy$

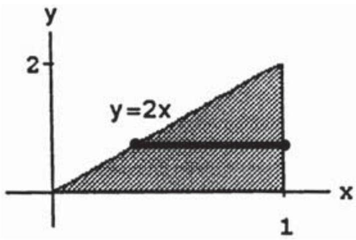
$= \pi \int_0^2 \left(16 - 16 + 8y^2 - y^4\right) dy = \pi \int_0^2 \left(8y^2 - y^4\right) dy = \pi \left[\frac{8}{3}y^3 - \frac{y^5}{5}\right]_0^2 = \pi \left(\frac{64}{3} - \frac{32}{5}\right) = \frac{224\pi}{15}$

46. (a) $r(y) = 0$ and $R(y) = 1 - \frac{y}{2}$

$\Rightarrow V = \int_0^2 \pi \left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 \left(1 - \frac{y}{2}\right)^2 dy = \pi \int_0^2 \left(1 - y + \frac{y^2}{4}\right) dy$

$= \pi \left[y - \frac{y^2}{2} + \frac{y^3}{12}\right]_0^2 = \pi \left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$



(b) $r(y) = 1$ and $R(y) = 2 - \frac{y}{2}$

$\Rightarrow V = \int_0^2 \pi \left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[\left(2 - \frac{y}{2}\right)^2 - 1\right] dy = \pi \int_0^2 \left(4 - 2y + \frac{y^2}{4} - 1\right) dy$

$= \pi \int_0^2 \left(3 - 2y + \frac{y^2}{4}\right) dy = \pi \left[3y - y^2 + \frac{y^3}{12}\right]_0^2 = \pi \left(6 - 4 + \frac{8}{12}\right) = \pi \left(2 + \frac{2}{3}\right) = \frac{8\pi}{3}$

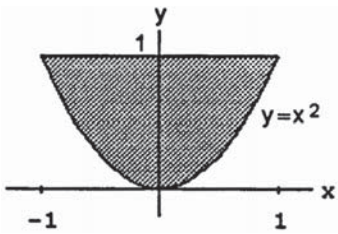
47. (a) $r(x) = 0$ and $R(x) = 1 - x^2$

$\Rightarrow V = \int_{-1}^1 \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{-1}^1 \left(1 - x^2\right)^2 dx = \pi \int_{-1}^1 \left(1 - 2x^2 + x^4\right) dx$

$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^1 = 2\pi \left(1 - \frac{2}{3} + \frac{1}{5}\right)$

$= 2\pi \left(\frac{15-10+3}{15}\right) = \frac{16\pi}{15}$



(b) $r(x) = 1$ and $R(x) = 2 - x^2 \Rightarrow V = \int_{-1}^1 \pi \left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^1 \left[\left(2 - x^2\right)^2 - 1\right] dx$

$= \pi \int_{-1}^1 \left(4 - 4x^2 + x^4 - 1\right) dx = \pi \int_{-1}^1 \left(3 - 4x^2 + x^4\right) dx = \pi \left[3x - \frac{4}{3}x^3 + \frac{x^5}{5}\right]_{-1}^1 = 2\pi \left(3 - \frac{4}{3} + \frac{1}{5}\right)$

$= \frac{2\pi}{15}(45 - 20 + 3) = \frac{56\pi}{15}$

(c) $r(x) = 1 + x^2$ and $R(x) = 2 \Rightarrow V = \int_{-1}^1 \pi \left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^1 \left[4 - \left(1 + x^2\right)^2\right] dx$

$= \pi \int_{-1}^1 \left(4 - 1 - 2x^2 - x^4\right) dx = \pi \int_{-1}^1 \left(3 - 2x^2 - x^4\right) dx = \pi \left[3x - \frac{2}{3}x^3 - \frac{x^5}{5}\right]_{-1}^1 = 2\pi \left(3 - \frac{2}{3} - \frac{1}{5}\right)$

$= \frac{2\pi}{15}(45 - 10 - 3) = \frac{64\pi}{15}$